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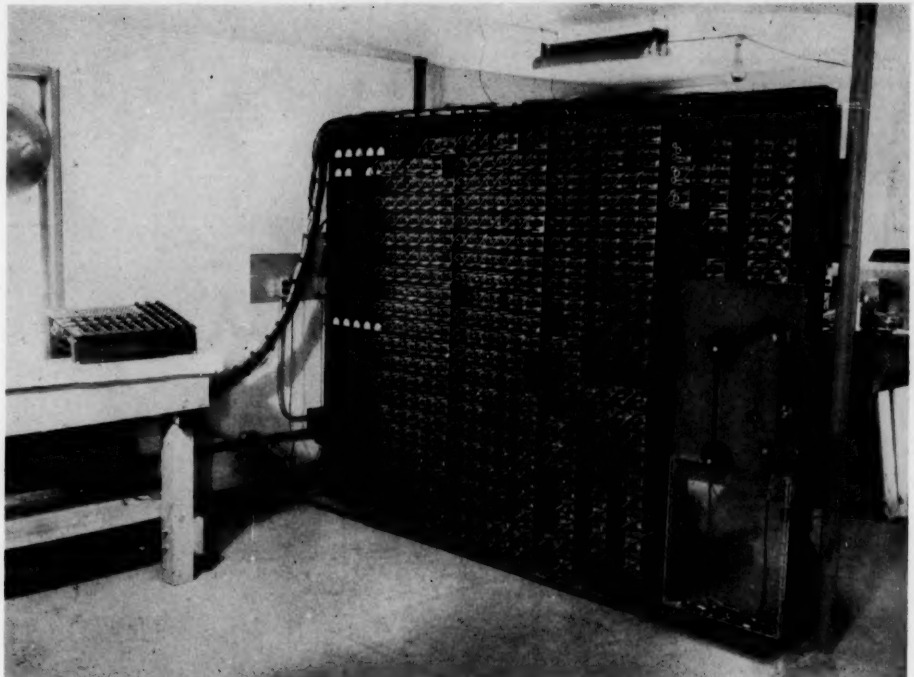

THE HALLER, RAYMOND AND BROWN SPECIAL PURPOSE COMPUTER

Stability Conditions in the Numerical Treatment of Parabolic Differential Equations

1. Introduction. The numerical solution of a hyperbolic partial difference equation is in some circumstances subject to an "instability" which has been shown by Lewy and others[1] to have a simple significance with reference to the differential equation which the difference equation approximates. For example, the difference equation

$$(1) \qquad (\phi_{n,j+1} - 2\phi_{n,j} + \phi_{n,j-1})/(\Delta t)^2 = c^2(\phi_{n+1,j} - 2\phi_{n,j} + \phi_{n-1,j})/(\Delta x)^2$$

may be used to approximate the wave equation

$$(2) \qquad \phi_{tt} = c^2\phi_{xx}.$$

If initial values, $\phi_{n,0}$ and $\phi_{n,1}$, are specified (corresponding to the specification of $\phi(x,0)$ and $\phi_t(x,0)$), equation (1) may be used to evaluate first $\phi_{n,2}$, then $\phi_{n,3}$, etc. If a "component" of $\phi_{n,j}$, i.e. a perturbation to $\phi_{n,j}$ which preserves the satisfaction of the difference equation and its boundary specifications other than initial values, varies with n as $(-1)^n$ (i.e. with the shortest wavelength which can be represented with the lattice spacing Δx) and depends upon j as K^j, then from (1)

$$(3) \qquad K^2 - 2K + 1 = -4\sigma K; \quad \sigma = (c\Delta t/\Delta x)^2.$$

If $\sigma > 1$, equation (3) has one real root of magnitude greater than unity; hence this (shortest wavelength) component of ϕ grows exponentially with j. Thus if this component is minutely excited, say through rounding errors, its exponential growth may, for sufficiently large j, destroy the resemblance of $\phi_{n,j}$ to the solution of (2). The possibility of such an uncontrolled growth of error is termed "instability," its absence "stability." For $\sigma \leqslant 1$ no such instability occurs.

The significance of the abrupt change in behavior of (1) as $c\Delta t$ exceeds Δx may be seen as follows: Each $\phi_{n,j}$ is computed by (1) with the use of preceding values, $\phi_{n,j-1}$ and $\phi_{n,j-2}$, and the neighboring values, $\phi_{n-1,j-1}$ and $\phi_{n+1,j-1}$. Thus $\phi_{n,j}$ is subject to influence only from within a "generating cone" described by $j' < j$, $|n - n'| \leqslant |j - j'|$. It is ultimately determined by those initial values $\phi_{n',0}$ for which $|n' - n| \leqslant j$. In the difference equation the propagation of influences is thus limited to rates $\leqslant \Delta x/\Delta t$. The differential equation (2) represents the propagation of influence along the characteristic lines, $x \pm ct = \text{constant}$; hence cannot be well represented by (1) unless $\Delta x/\Delta t$ is at least as great as c. A more general study of hyperbolic difference equations shows the presence of instability whenever the mesh ratio $\Delta x/\Delta t$ is too small to permit the propagation of influence along the characteristics of the corresponding differential equation.

In the solution of parabolic difference equations stability conditions depending on the mesh intervals may also arise. They are considerably more burdensome than the above condition, usually requiring that Δt be small of

the order of (Δx).[2] By suitably modifying the difference equation, the limitation imposed by the stability condition can be removed. The character of the diffusion process described by a parabolic equation suggests that a stability condition which limits $\Delta t/(\Delta x)^2$ or even $\Delta t/\Delta x$ is not fundamental. The effects of a localized disturbance are appreciable within a range which increases with elapsed time, t, about as $t^{\frac{1}{2}}$. Thus the region of appreciable influence can asymptotically be included within an arbitrarily narrow generating cone.

In sections 2 and 3 the stability condition limiting the time interval in a conventional treatment of a simple parabolic differential equation is displayed and its burdensome character illustrated.

In section 4 an alternative simple parabolic difference equation approximating the same differential equation is shown to be stable for all values of $\Delta t/(\Delta x)^2$. It is shown in section 5 that the condition for convergence to the solution of the corresponding parabolic differential equation is $\Delta x \to 0$, $\Delta t/\Delta x \to 0$. In sections 7 and 8 various generalizations of this method of calculation are examined, particularly with reference to their stability. In these generalizations the definition of "instability" given above becomes inapplicable. The term will there be used in a looser sense to indicate a tendency toward the exaggeration of initially small "errors." Accordingly the indications of stability given there are plausibility-proofs rather than rigorous arguments.

2. Interval limitation for a one-dimensional diffusion equation. The diffusion equation

$$\phi_t = p(x)\phi_{xx}; \quad p(x) > 0 \tag{4}$$

may be approximated by the difference equation

$$(\phi_{n,j+1} - \phi_{n,j})/\Delta t = p_n(\phi_{n-1,j} - 2\phi_{n,j} + \phi_{n+1,j})/\Delta x^2; \quad p_n = p(n\Delta x). \tag{5}$$

If initial vaues $\phi_{n,0}$ are specified, equation (5) permits the successive evaluation of $\phi_{n,1}$, $\phi_{n,2}$, etc. An approximate stability condition for this process is readily derived by the assumption that p_n changes slowly with n. An alternating component of $\phi_{n,j}$, $\phi_{n,j} \sim (-1)^n$, then appears in $\phi_{n,j+1}$ modified by the factor $1 - 4\Delta t p_n/\Delta x^2$,

$$\phi_{n,j+1} \sim (1 - 4\sigma p_n)\phi_{n,j}; \quad \sigma = \Delta t/\Delta x^2.$$

An exponential increase of this alternating component may be expected if $p_n > 1/2\sigma$ over any appreciable region. The condition for stability is then

$$\sigma \leqslant 1/2p_M \tag{6}$$

where p_M approximates the maximum of the p_n. More precisely, the procedure described by (5) is unstable if an error in $\phi_{n,j}$ recurs in the same form in $\phi_{n,j+1}$ with increased magnitude, i.e. if an error component

$$\delta\phi_{nj} = K^j\lambda_n; \quad |K| > 1$$

satisfies (5) and λ_n satisfies homogeneous boundary conditions corresponding to those imposed on $\phi_{n,j}$. For definiteness it may be assumed that $\phi_{0,j}$ and $\phi_{N,j}$ are specified. Then an "error-eigenfunction," λ_n, must satisfy

$$-4p\lambda_n = (K-1)\lambda_n/\sigma = p_n(\lambda_{n-1} - 2\lambda_n + \lambda_{n+1}); \quad \lambda_0 = \lambda_N = 0. \tag{7}$$

Multiplying by λ_n/p_n and summing yields

$$-4p\sum_{n=0}^{N}\lambda_n^2/p_n = \sum_{n=0}^{N}[\lambda_n(\lambda_{n+1}-\lambda_n)-\lambda_n(\lambda_n-\lambda_{n-1})]$$

$$= \sum_{n=0}^{N-1}\lambda_n(\lambda_{n+1}-\lambda_n)-\sum_{n=0}^{N-1}\lambda_{n+1}(\lambda_{n+1}-\lambda_n)$$

$$= -\sum_{n=0}^{N-1}(\lambda_{n+1}-\lambda_n)^2 < 0;$$

thus the positiveness of an eigenvalue, p, follows from the assumed positiveness of p_n. This requires that $K < 1$; hence instability can arise only if for some error-eigenfunction $K < -1$. The stability condition (6) holds strictly if p_M denotes the greatest eigenvalue, p, of (7).

The stability condition (6) imposes a burdensome limitation on the numerical treatment of parabolic differential equations by this method since a moderately fine spatial division may require an immoderately fine temporal division. This will particularly be the case if a very fine structural representation is required (large N) or if a great disparity exists between the largest and smallest values of p_n. If the range in t for which a solution is required is of the order of the time required for effective diffusion to the distance $N\Delta x$, the number of time intervals required will be of the order of

$$t_{\max}/\Delta t \sim N^2 p_M/p_0$$

where p_0 approximates the smallest of the p_n.

3. Application to the radial diffusion equation. To illustrate the above general conditions we consider the simple diffusion equation

$$\phi_t = \phi_{rr} + \phi_r/r; \qquad \begin{aligned}\phi(r,0) &= \Phi(r)\\ \phi(1,t) &= a\\ \phi(R,t) &= b.\end{aligned} \tag{8}$$

Writing $r = e^x$ puts (8) in the form

$$\phi_t = e^{-2x}\phi_{xx}; \qquad \begin{aligned}\phi(x,0) &= \Phi(x)\\ \phi(0,t) &= a\\ \phi(\ln R,t) &= b.\end{aligned} \tag{9}$$

This is of the form of equation (4) with $p(x) = e^{-2x}$. The error-eigenfunction equation (7) is now

$$4p\eta_n = (1-K)\eta_n/\sigma = e^{-2n\Delta x}(\eta_{n+1}+2\eta_n+\eta_{n-1}); \quad \eta_0 = \eta_N = 0, \tag{10}$$

where $\eta_n = (-1)^n\lambda_n$.

The greatest eigenvalue, p_M, is readily approximated by a variation principle:

For each solution of (10) the expression

$$\chi = 2\sum_{n=0}^{N-1}\eta_n(\eta_n+\eta_{n+1})/\sum_{0}^{N}\eta_n^2 e^{2n\Delta x}$$

takes on a stationary value, namely $4p$.

The maximum value permitted to χ is then

(11) $$\chi_M = 4p_M = (1 - K_{\min})/\sigma.$$

The condition for stability, $K_{\min} \geqslant -1$, then requires

$$\sigma \leqslant 2/\chi_M \equiv \sigma_{\text{crit}}.$$

A lower bound to χ_M, hence an upper bound to σ_{crit} (for $N = \infty$) has been obtained by setting $\eta_n = ne^{-\beta n}$ and maximizing χ with respect to β. The resulting approximation to σ_{crit} is displayed (*vs.* Δx) in Fig. 1. To estimate the error in this approximation numerical solutions of the difference equations for two values of Δx were carried out with excessive σ's. The rates of growth of the dominant instabilities determine χ_M (by eq. 11), hence σ_{crit}. These two "experimental" values are also shown in Fig. 1.

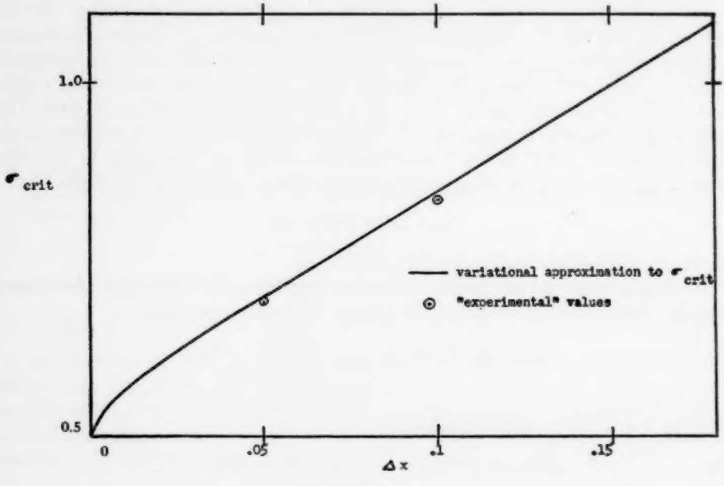

FIG. 1.

To illustrate the uncomfortable limitation imposed by the stability condition we assume (arbitrarily) that the calculation is to be carried to sufficiently great t that $\phi(r,t)$ at $r = 100$ is substantially affected by the inner boundary value, a. This requires $t_{\max} \sim (100)^2$. To represent adequately the structure of ϕ near $r = 1$ requires, say, $\Delta x = 0.1$. Then the stability condition requires

$$\Delta t \leqslant \sigma_{\text{crit}}(\Delta x)^2 = 0.0085.$$

The number of "cycles" (j-values) required for the calculation is then $t_{\max}/\Delta t \sim 10^6$. A moderately fast electronic digital computer could complete the calculation in a few hundred hours!

4. A generally stable difference equation. The difference equation (5) determines a set of numbers, ϕ_{nj}, which approximate the solution of the

differential equation (4) at the junction points of the rectangular lattice $x = n\Delta x, t = j\Delta t$. The alternative difference equation here considered makes use of a diagonal lattice, obtained by omitting those junction points for which $n + j$ is, say, odd. The term ϕ_t is now represented by the difference of two ϕ_{nj}-values of the same n and j's differing by 2; i.e. $\phi_t(n\Delta x, j\Delta t) \sim (\phi_{n,j+1} - \phi_{n,j-1})/2\Delta t; n + j$ odd. In the representation of ϕ_{xx} by a second difference with respect to n the end terms, $\phi_{n-1,j}$ and $\phi_{n+1,j}$, may be used as previously but the term, $\phi_{n,j}$, corresponds to a point omitted from the lattice. It is therefore replaced by the mean of the two terms of neighboring j-values.

$$\phi_{xx}(n\Delta x, j\Delta t) \sim (\phi_{n-1,j} - \phi_{n,j-1} - \phi_{n,j+1} + \phi_{n+1,j})/(\Delta x)^2; \quad n + j \text{ odd.}$$

The resulting difference approximation to (4) is

$$(12) \quad \phi_{n,j+1} - \phi_{n,j-1} = 2p_n\sigma(\phi_{n-1,j} - \phi_{n,j-1} - \phi_{n,j+1} + \phi_{n+1,j}),$$
$$n + j \text{ odd}, \quad p_n > 0, \quad \sigma = \Delta t/\Delta x^2$$

which may be written as

$$(13) \qquad \phi_{n,j+1} = \phi_{n,j-1} + \alpha_n(\phi_{n-1,j} - 2\varphi_{n,j-1} + \phi_{n+1,j}),$$

where

$$\alpha_n = 2p_n\sigma/(1 + 2p_n\sigma); \quad 0 < \alpha_n < 1.$$

If the differential equation has terms in ϕ_x and ϕ these may be represented by $(\phi_{n+1,j} - \phi_{n-1,j})/2\Delta x$ and $(\phi_{n,j+1} + \phi_{n,j-1})/2$ respectively.

It is to be noted that the initial conditions required to permit calculation with (13) require the specification of ϕ-values for two initial times (j-values), or initial boundaries, $x(t)$, when only one would be required by the conventional difference equation. This requirement suggests that (13) is of hyperbolic rather than parabolic character (cf. section 5). The second set of initial values may be computed from the first set by use of the conventional equation with sufficiently small Δt.

In each cycle of the calculation $\phi_{n,j}$ may be evaluated for all n-values associated with one j-value, even n occurring in one cycle, odd n in the next. Alternatively the $\phi_{n,j}$ may be evaluated in each cycle for all n along a diagonal line $n + j =$ constant (or $n - j =$ constant). These two orderings of the calculation, called the "leap-frog" and "pyramid" methods respectively, are shown in Fig. 2.

In the pyramid method the cycle number may be used to index the ϕ-values in place of j. If the cycle number, m, is written as a super-script, the difference equation (13) takes the form

$$(14) \qquad \phi_n^{m+1} = \phi_n^m + \alpha_n(\phi_{n-1}^{m+1} - 2\phi_n^m + \phi_{n+1}^m).$$

This form of the difference equation displays clearly the similarity to the "extrapolated Liebmann" form of the relaxation method.[2]

The qualitative behavior of the solution of (13) may be investigated by examining the propagation of a spatially sinusoidal component of φ, treating p_n, hence also α_n, as a constant. If

$$\lambda_{n,j} = K^j e^{in\gamma}$$

satisfies (13) for constant α_n then

$$K = K^{-1} + \gamma(e^{-i\gamma} - 2K^{-1} + e^{i\gamma}).$$

Hence

$$K^2 - 2K\alpha\cos\alpha + (2\alpha - 1) = 0. \tag{15}$$

For each γ, two values of K satisfy (15). Thus a sinusoidal component of ϕ has two modes of propagation with increasing t. The propagation factors, K, are shown in Fig. 3. For γ near 0 or near π the propagation factors are

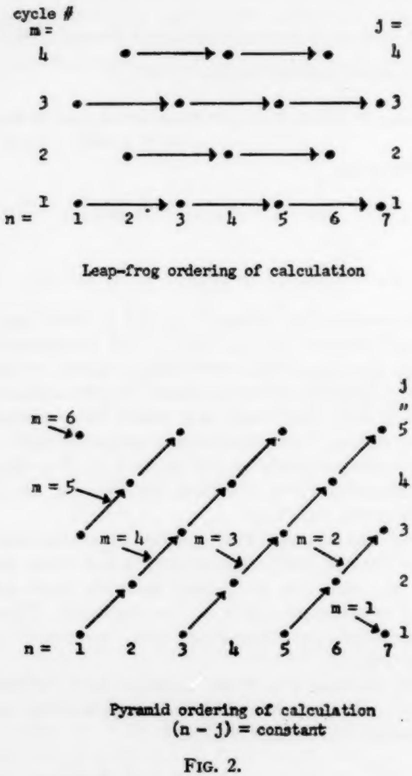

FIG. 2.

real, for intermediate γ they are complex. In no case is the magnitude of K greater than unity. The difference equation (13) is thus seen to be stable for any constant positive value of $p_n\sigma$.

To show the stability of (13) for a varying α_n, we assume the presence of an error-eigenfunction of the form

$$\phi_{n,j} = K^j\lambda_n; \quad \lambda_0 = \lambda_N = 0.$$

Substitution in (13) yields

$$\lambda_{n-1} + \lambda_n(1 - K^2 - 2\alpha_n)/K\alpha_n + \lambda_{n+1} = 0$$

or

$$(16) \quad \omega^2(2\lambda_n - \lambda_{n-1} - \lambda_{n+1}) - 4\omega(\alpha_n^{-1} - 1)\lambda_n + (2\lambda_n + \lambda_{n-1} + \lambda_{n+1}) = 0,$$

where

$$(17) \qquad K = (\omega - 1)/(\omega + 1).$$

Multiplying (16) by λ_n^* and summing over $n = 1,2,\cdots, N-1$ produces, after slight rearrangement of terms, the equation

$$(18) \quad \omega^2 \sum_1^N |\lambda_n - \lambda_{n-1}|^2 - 4\omega \sum_1^{N-1} (\alpha_n^{-1} - 1)|\lambda_n|^2 + \sum_1^N |\lambda_n + \lambda_{n-1}|^2 = 0.$$

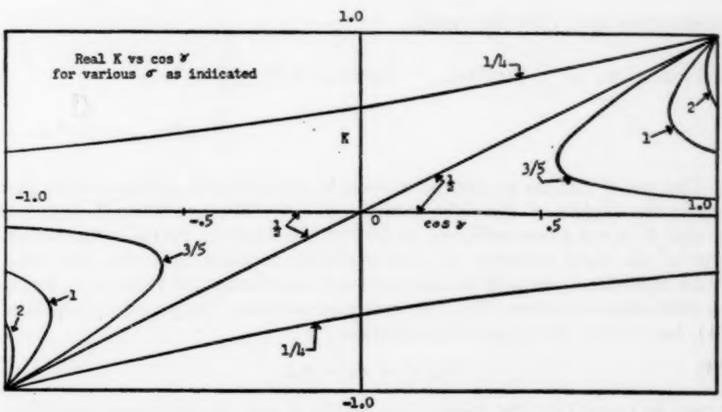

FIG. 3.

The positiveness of the three summations ensures that the real part of ω is positive, hence by (17)

$$(19) \qquad |K| < 1.$$

If the fixed end conditions, $\lambda_0 = \lambda_n = 0$, are replaced by the homogeneous boundary conditions,

$$\begin{aligned} \lambda_1 - \lambda_0 &= A\lambda_0 \\ \lambda_{N-1} - \lambda_N &= B\lambda_N \end{aligned} \qquad A \text{ and } B \text{ real and positive,}$$

equation (18) is modified by the inclusion of additional positive terms in the first and third summations. Then again (19) follows from the positiveness of $(\alpha_n^{-1} - 1)$; hence the difference equation (13) is stable for any positive σ.

5. Dependence of the solution on the lattice spacings. To illustrate the dependence of the difference equation (13) and its solution on the lattice intervals, Δx and Δt, we examine in detail the special case of constant $p(x)$.

This constant may be given the value unity without loss of generality. The difference equation is then

$$\text{(20)} \quad \phi_{n,j+1} - \phi_{n,j-1} = 2\sigma\,(\phi_{n-1,j} - \phi_{n,j-1} - \phi_{n,j+1} + \phi_{n+1,j}),\ \sigma = \Delta t/\Delta x^2,$$

which approximates the homogeneous, one-dimensional diffusion equation

$$\text{(21)} \qquad \phi_t = \phi_{xx}.$$

If the set of four ϕ-values connected by (20) are assumed to be imbedded in a function $\phi(x,t)$ which permits a power series expansion about the "center point" (n,j), they may be described by the expansions

$$\phi_{n\pm1,j} = \phi \pm \Delta x\phi_x + \frac{1}{2}(\Delta x)^2\phi_{xx} \pm \frac{1}{6}(\Delta x)^3\phi_{xxx} + \frac{1}{24}(\Delta x)^4\phi_{xxxx} + \cdots,$$

$$\phi_{n,j\pm1} = \phi \pm \Delta t\phi_t + \frac{1}{2}(\Delta t)^2\phi_{tt} \pm \frac{1}{6}(\Delta t)^3\phi_{ttt} + \frac{1}{24}(\Delta t)^4\phi_{tttt} + \cdots.$$

Substitution into (20) then yields

$$\text{(22)} \quad \phi_t - \phi_{xx} = \frac{1}{12}(\Delta x)^2\phi_{xxxx} - (\Delta t)^2\phi_{tt}/(\Delta x)^2 - \frac{1}{6}(\Delta t)^2\phi_{ttt} - \frac{1}{12}(\Delta t)^4\phi_{tttt}/(\Delta x)^2 + \cdots.$$

The use of (20) as an approximation to the diffusion equation thus consists in the neglect of the right member of (22). The use of small values of Δx and Δt is not alone sufficient to justify this neglect. To make the second term of the right member of (22) negligible requires also that the ratio, $\Delta t/\Delta x$ be small. If Δx and Δt approach zero with constant ratio, $c = \Delta x/\Delta t$, the difference equation (20) does not approximate the parabolic equation (21), but rather the hyperbolic equation

$$\text{(23)} \qquad \phi_{tt}/c^2 + \phi_t = \phi_{xx}.$$

It may be noted that the mesh ratio, $\Delta t/\Delta x$, is just the maximum permitted by the stability condition for the treatment of this hyperbolic equation by the usual difference method. The stability of the difference equation (20) may be regarded as arising from the introduction of this hyperbolic term in the approximating differential equation.

The above considerations indicate that the requirement for accuracy of treatment of the parabolic equation approximated by (20) is the smallness of both Δx and $\Delta t/\Delta x$. The ratio, $\Delta t/\Delta x$ is not, however, required to be small in the order of Δx, as would be needed for a fixed value of σ. Since (21) implies

$$\phi_{tt} = \phi_{xxxx}$$

the dominant terms in the right member of (22) may be written

$$\phi_t - \phi_{xx} = \frac{1}{12}(\Delta x)^2\phi_{xxxx}(1 - 12\sigma^2) + \cdots.$$

Thus for fixed Δx the optimum value for Δt, from considerations of accuracy alone, lies near $12^{-\frac{1}{2}}\Delta x^2$.

The accuracy of the difference equation representation of (21) may be examined by reference to the Fourier expansion used in establishing its stability. A component of ϕ having the sinusoidal dependence, $e^{in\gamma}$, is exponentially attenuated with increasing j by the factor K per cycle. The attenuation factor, K, is determined by (15) and is double valued (again suggesting the hyperbolic character of the difference equation). As shown in Fig. 4, K_+, the greater value of K, closely approximates the correct attenuation factor, $K_e = e^{-\gamma^2\sigma}$ (i.e. the factor by which a sinusoidal component $e^{i\gamma x/\Delta x}$ is attenuated in time Δt by the differential equation (21)) for small γ. With increasing γ this greater K becomes progressively more unrealistic,

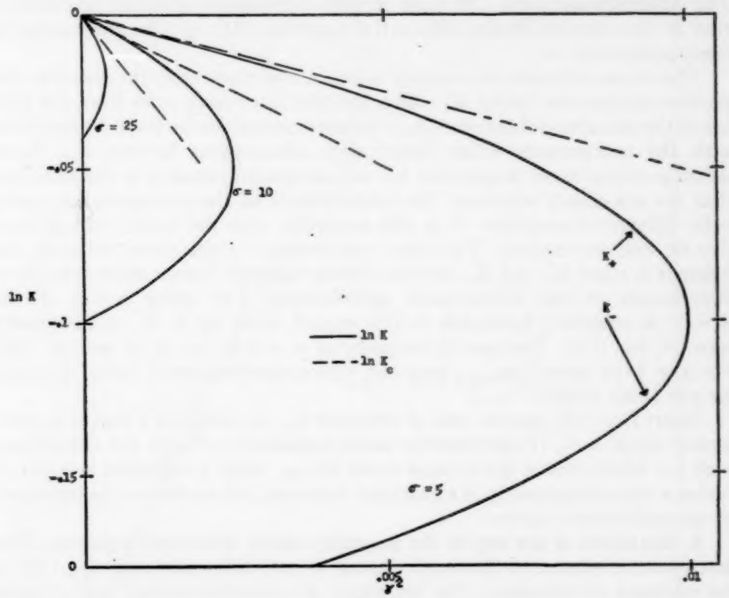

FIG. 4.

the more rapidly the greater is σ. For

$$0 \leqslant \cos\gamma < (1 - 1/4\sigma^2)^{\frac{1}{2}}$$

the two K's become complex, of common magnitude $[(2\sigma - 1)/(2\sigma + 1)]^{\frac{1}{2}}$. For negative values of $\cos\gamma$ the attenuation factors are the negatives of those for the corresponding positive $\cos\gamma$. Since only even values of $n + j$ are considered these negative values of $\cos\gamma$ are superrogatory.

The rapid increase of the error in K_+ with increasing γ would seem to indicate that the difference equation (20) is of unacceptable accuracy except for small σ. After a large number of cycles of calculation (large j), however, the two values, K_+ and K_e, appear in the solutions of (20) and (21) raised to the power j. For sufficiently large j, both $K_+{}^j$ and $K_e{}^j$ are essentially zero

except for very small γ. An approximate measure of error is thus the greatest value of the discrepancy,

$$D_j = K_e{}^j - K_+{}^j = e^{-j\sigma\gamma^2} - K_+{}^j.$$

The solution of (15) in power series in γ^2 yields

$$K_+ = K_e \exp[-\gamma^4(\sigma^3 - \sigma/12) + \cdots].$$

For $j \gg 4(\sigma - 1/12\sigma)$ the greatest discrepancy occurs at $\gamma^2 \cong 2/j\sigma$ and has the value

$$D_j \cong 4e^{-2}(\sigma - 1/12\sigma)/j.$$

The difference equation (20) thus permits increasingly accurate representation of the solution of the differential equation (21) as j becomes large in comparison with 4σ.

The above estimate of accuracy takes into account only the errors in the greater attenuation factor K_+. An additional error may arise from the failure of the initial specification of $\phi_{n,j}$-values to associate the correct amplitude with the components which decay with attenuation factors, K_+. Since initial ϕ-values must be specified for two successive j-lines it is not sufficient that the ϕ's closely represent the initial values of the corresponding parabolic differential equation. It is also necessary that the initial time derivative be well represented. The latter requirement is the more stringent the larger is σ, since K_+ and K_- are then closer together, hence more difficult to discriminate in the initial-value specification. The value $\sigma = \frac{1}{2}$ (hence $\alpha = \frac{1}{2}$) is especially favorable in this regard, since for it K_- is identically zero (cf. eq. (15)). The special property of $\alpha = \frac{1}{2}$ is also to be seen in (13). For $\alpha = \frac{1}{2}$ the terms in $\phi_{n,j-1}$ drop out, hence specification of initial ϕ-values for one j-line suffices.

Apart from the special case of constant p_n, no choice of σ makes α_n uniformly equal to $\frac{1}{2}$. It nevertheless seems expedient to begin the calculation with a σ which makes $\frac{1}{2}$ a typical value for α_n. After a sufficient number of cycles σ can conveniently (i.e. without requiring interpolation) be increased by an odd integral factor.

6. Examples of the use of the generally stable difference equation. The first example chosen to illustrate the use of the difference equation (13) is the equation of diffusion to the boundary of a one-dimensional semi-infinite medium. Its differential equation and boundary conditions are

$$\phi_t = \phi_{xx} \text{ for } t > 0, \quad 0 < x < \infty,$$
$$\phi(x,0) = 1, \quad \phi(0,t) = 0.$$

It has the simple solution

$$\phi(x,t) = \operatorname{erf}(x/2t^{\frac{1}{2}}).$$

Numerical solutions were obtained by the pyramid method for two values of σ, $\frac{1}{2}$ and 5. The boundary values are

$$\phi_0{}^m = 0, \quad \phi_n{}^0 = 1 \text{ for } n > 0. \tag{24}$$

Subsequent ϕ-values are determined by the relation

$$\phi_n^{m+1} = \phi_n^m + \alpha_n(\phi_{n+1}^{m+1} - 2\phi_n^m + \phi_{n-1}^m), \tag{25}$$
$$\alpha_n = \tfrac{1}{2} \text{ or } 10/11 \text{ as } \sigma = \tfrac{1}{2} \text{ or } 5.$$

For each cycle (m-value) the ϕ_n^{m+1} are computed for successively smaller n, beginning with an n sufficiently large that ϕ_n^{m+1} does not differ appreciable from unity. (Here $2m = n + j$.)

The resulting ϕ-values are displayed in Fig. 5 and 6, plotted for several values of $t = \Delta t(2m - n)$ vs. the similarity variable,

$$\mu = x/2t^{\frac{1}{2}} = n/2[\sigma(2m - n)]^{\frac{1}{2}}.$$

The approach of $\phi_n{}^m$ to $\phi(\mu)$ with increasing $j = 2m - n$, hence *a fortiori* with increasing $t = \sigma j \Delta x^2$, is noticeably more rapid for the smaller σ.

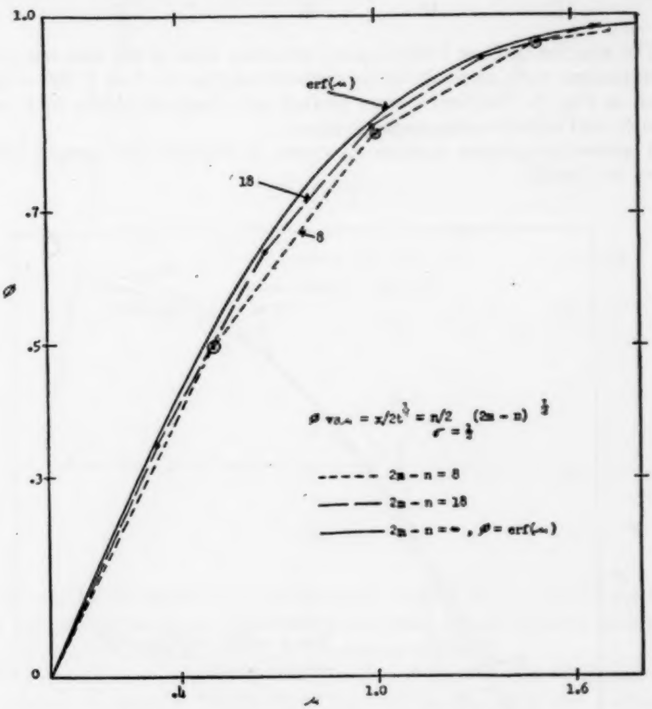

FIG. 5. Results of One Dimensional Diffusion Problem

A second example used is the radial diffusion equation,

$$\phi_t = \phi_{rr} + \phi_r/r; \quad r \geq 1. \tag{26}$$

Setting $r = e^x$ permits writing (26) in the form

$$\phi_t = e^{-2x}\phi_{xx}; \quad x \geq 0. \tag{27}$$

The boundary values

$$\phi(0,t) = 0, \quad \phi(x,0) = 1$$

were chosen.

Again the pyramid ordering was used; hence the difference equation is (25) with

$$\alpha_n = 2\sigma e^{-2n\Delta x}/(1 + 2\sigma e^{-2n\Delta x}).$$

The boundary conditions of the difference equation are given again by (24). Three solutions were obtained, with intervals as follows:

Case	Δx	Δt	$\sigma = \Delta t/\Delta x^2$
I	.05	.00125	$\frac{1}{2}$
II	.05	.0125	5
III	.10	.05	5

The solution in Case I shows good accuracy even in the first few cycles. A comparison with an analytically derived solution for $t = 1/100 = 8\Delta t$ is shown in Fig. 7. The four points plotted are obtained in the fifth, sixth, seventh, and eighth cycles respectively.

A somewhat greater number of cycles is required for comparable accuracy in Case II.

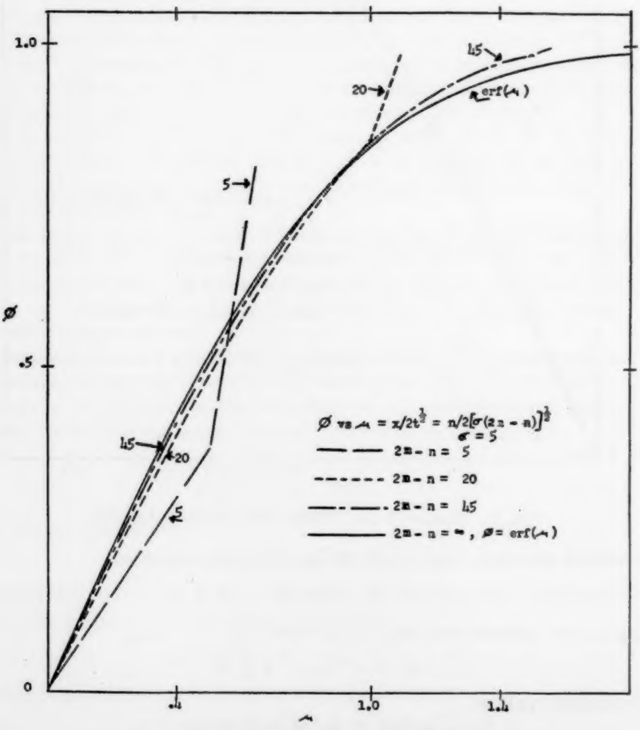

FIG. 6. Results of One Dimensional Diffusion Problem

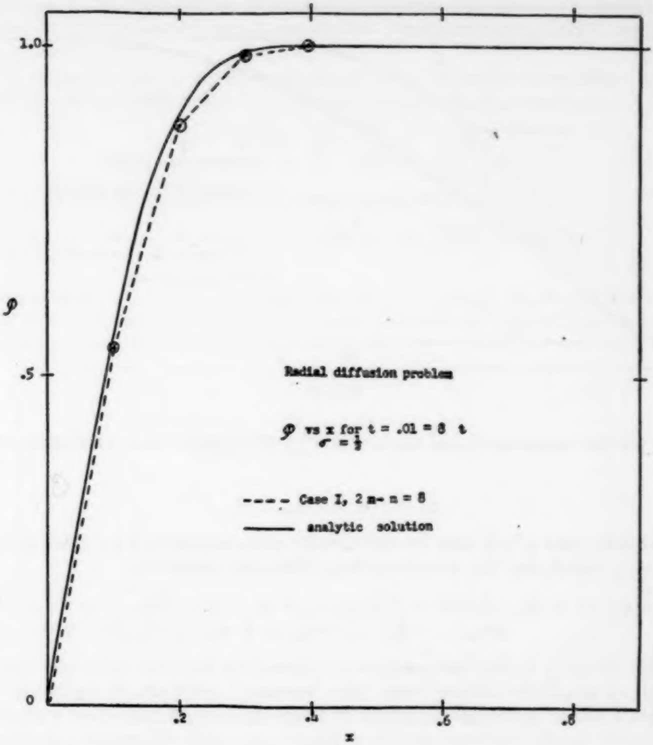

FIG. 7.

Its solution is shown for $t = 5\Delta t$ and $t = 17\Delta t$ in Fig. 8. For the latter time the plotted points are obtained in the ninth and subsequent cycles, and are of about the same accuracy as the points in Fig. 7.

The solution in Case III is shown in Fig. 9 for $t = 5\Delta t$, $15\Delta t$, and $40\Delta t$. The values for $40\Delta t = 2.0$ (derived from the twenty-first and subsequent cycles) are not distinguishable in this graph from the previous cases nor from the analytic solution.

7. Extension to other parabolic equations. The demonstration given above of the stability of the leap-frog (or pyramid) method depends upon an eigenfunction expansion of the errors in the dependent variable. (So, in fact, does the definition of stability there used.) To justify the use of these methods for broader classes of parabolic equations other indications of stability must be sought. In section 8 several types of linear parabolic difference equations are shown to display properties indicative of stability. No uniform definition and proof of stability has been produced, however.

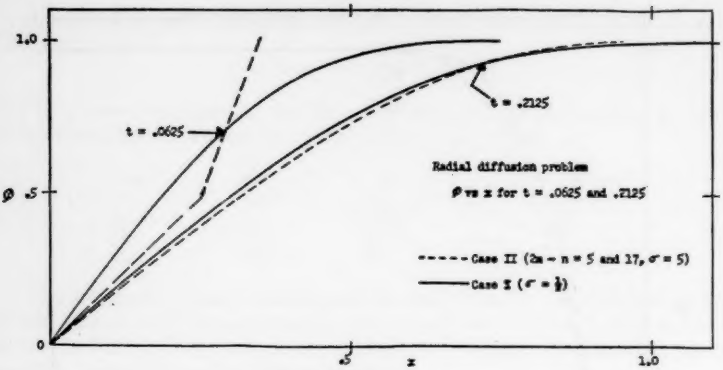

Fig. 8.

A similar computational technique can be applied to a nonlinear equation,

(28) $$\phi_t = f(\phi,\ \phi_x,\ \phi_{xx},\ x,\ t).$$

In suitable cases $\phi(x,t)$ may be sufficiently well represented by a set of numbers, $\phi_{n,j}$, satisfying the corresponding difference equation,

(29) $$\begin{aligned}(\phi_{n,j+1} - \phi_{n,j-1})/2\Delta t = f[\tfrac{1}{2}(\phi_{n,j+1} + \phi_{n,j-1}),\ \ \tfrac{1}{2}(\phi_{n+1,j} - \phi_{n-1,j})/\Delta x,\\ (\phi_{n+1,j} - \phi_{n,j-1} - \phi_{n,j+1} + \phi_{n-1,j})/(\Delta x)^2,\ \ n\Delta x,\ \ j\Delta t].\end{aligned}$$

The stability of this computational procedure depends upon the rates of growth of small deviations from this "correct" solution. If equation (29) permits a power series development in these deviations then they will, when sufficiently small, be governed by a linear parabolic difference equation of

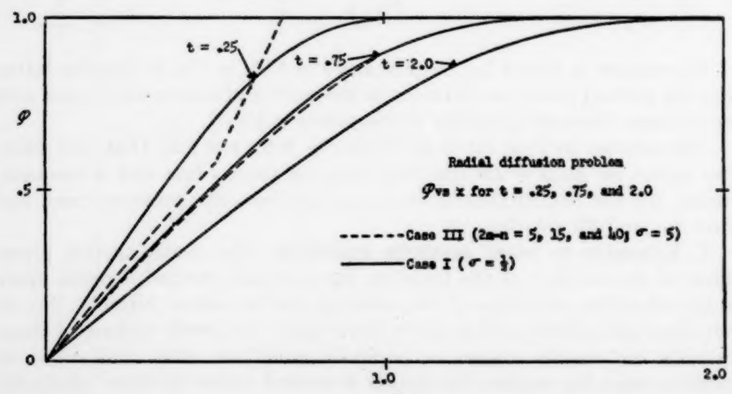

Fig. 9.

the same form. The above stability arguments for linear equations thus indicate a wide range of usefulness for this procedure as applied to nonlinear equations.

To illustrate this procedure we consider the equation describing the one-dimensional isothermal flow of a perfect gas in a porous medium.

$$\phi_t = (\phi^2)_{xx} \quad \text{for} \quad t > 0, \quad x > 0 \qquad (30)$$
$$\phi(x,0) = 1; \quad \phi(0,t) = 0.$$

A convenient corresponding difference equation is

$$\Delta\phi \equiv \phi_{n,j+1} - \phi_{n,j-1} = 2\sigma[\phi^2_{n-1,j} - 2\phi^2_{n,j-1} + \phi^2_{n+1,j} - \Delta\phi(\phi_{n-1,j} + \phi_{n+1,j})].$$

The use of $\phi_{n-1,j} + \phi_{n+1,j}$ rather than $\phi_{n,j-1} + \phi_{n,j+1}$ as the coefficient of $\Delta\phi$ in the right member is consistent with the order of approximation to (30)

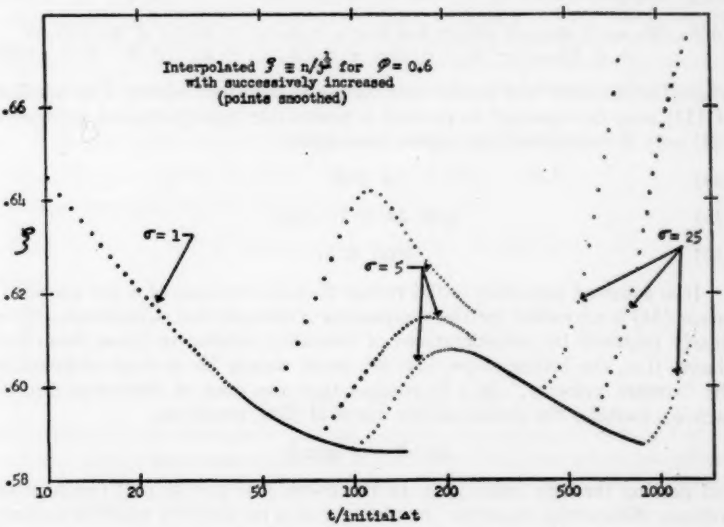

FIG. 10.

used and simplifies the explicit expression for $\phi_{n,j+1}$. This is

$$\phi_{n,j+1} = \phi_{n,j-1} + 2\sigma[\phi^2_{n-1,j} - 2\phi^2_{n,j-1} + \phi_{n+1,j}] \,/[1 + \sigma(\phi_{n-1,j} + \phi_{n+1,j})]. \qquad (31)$$

Solutions to (31) were obtained with σ initially given the value 1, later increased by successive factors of five. (An attempt to start a solution with $\sigma = 5$ led to uncontrolled oscillation.) When σ was not increased too soon, qualitatively correct solutions were obtained. The solution of (30) is of the form $\varphi(x/t^{\frac{1}{2}})$. Accordingly, to provide a measure of accuracy of the approximation of $\phi_{n,j}$ to the solution of (30), the $\phi_{n,j}$- values for each cycle were used to determine a graphically interpolated value of $\xi = n/j^{\frac{1}{2}}$ corresponding

to the arbitrarily selected value, $\phi_{n,j} = 0.6$. The discrepancy between ξ and its asymptote, approximately 0.586, indicates the order of inaccuracy of the solution. Fig. 10 shows values of ξ for a family of solutions of (31) differing in the number of cycles carried out at each value of σ. The abscissa, displayed logarithmically, shows t in units of the initial Δt rather than the number of cycles of calculation performed. The error (of approximation to the solution of (30)) is seen to increase and later subside slowly after each increase in σ. The peak error decreases rapidly with deferral of the increase in σ. To keep the error of the order of one percent requires a few hundred cycles at each σ-value, somewhat more in the later stages.

8. Stability for the general parabolic linear equation. The differential equation,

$$\phi_t = p(x,t)\phi_{xx} + q(x,t)\phi_x + r(x,t)\phi \tag{32}$$

may be approximated by the difference equation

$$\begin{aligned}(\phi_{n,j+1} - \phi_{n,j-1})/2\Delta t = p_{nj}(\phi_{n+1,j} - \phi_{n,j+1} - \phi_{n,j-1} + \phi_{n-1,j})/\Delta x^2 \\ + q_{nj}(\phi_{n+1,j} - \phi_{n-1,j})/2\Delta x + r_{nj}(\phi_{n,j+1} + \phi_{n,j-1})/2; \quad n + j \text{ odd},\end{aligned} \tag{33}$$

where the notation and lattice structure is as described above. The solution of (33) may be expected to provide a reasonable approximation to that of (32) only if throughout the region considered

$$p \geqslant 0 \tag{34}$$

$$|q\Delta t/\Delta x| \leqslant 1, \quad \text{and} \tag{35}$$

$$|r\Delta t| \ll 1. \tag{36}$$

It is assumed here that initial rather than final values of ϕ are specified; hence (34) is necessary for the uniqueness of the solution. Condition (35) is clearly required by considerations of causality similar to those described above. (i.e., the lattice slope, $|\Delta x/\Delta t|$, must clearly be at least as great as the "stream velocity," $|q|$.) It ensures that the cone of determination of each ϕ_{nj} includes the characteristic curve of (32), satisfying

$$dx/dt = - q(x,t)$$

and passing through $(n\Delta x, j\Delta t)$. In the event $p = q = 0$, (32) becomes an ordinary differential equation (involving x as a parameter) which is reasonably approximated by (33) only if (36) is satisfied.

We first consider the case $q \equiv r \equiv 0$. Equation (33) then reduces to

$$(\phi_{n,j+1} - \bar{\phi}_{nj}) = (\phi_{n,j-1} - \bar{\phi}_{nj})(1 - 2p_{nj}\sigma)/(1 + 2p_{nj}\sigma) \tag{37}$$

where

$$\bar{\phi}_{nj} = \tfrac{1}{2}(\phi_{n-1,j} + \phi_{n+1,j}) \quad \text{and} \quad \sigma = \Delta t/\Delta x^2 \tag{38}$$

hence

$$|\phi_{n,j+1} - \bar{\phi}_{nj}| \leqslant |\phi_{n,j-1} - \bar{\phi}_{nj}|. \tag{39}$$

Each stage of the computation is characterized by a "front" in the n,j lattice bounding the region for which ϕ-values have been computed. The front is specified by a single-valued function, $j(n)$, ($j + n$ even), satisfying

$j(n+1) = j(n) \pm 1$. For $j \leqslant j(n)$, ϕ_{nj} has been computed, otherwise not. In each step of the calculation a "low" point of the front, $j(n) = j(n \pm 1) - 1$, is replaced by $j(n) + 2$. Thus a "valley" in the front is replaced by a "hill." We define a function of the front consisting of the sum of squares of the first difference of ϕ, taken along the front:

$$F = \sum_n (\phi_{n, j(n)} - \phi_{n-1, j(n-1)})^2. \tag{40}$$

In a step of the calculation two terms of F are altered as a result of the replacement of $\phi_{n, j(n)}$ by $\phi_{n, j(n)+2}$. With the use of (39) it can be shown that this replacement diminishes F unless $p_{nj} = 0$ or $\phi_{n, j+1} = \bar{\phi}_{n, j}$, in which event F is unchanged. Thus as the calculation proceeds F decreases monotonically except by reason of influences introduced at the boundaries. This non-increasing character of the (positive) function, F, may be regarded as indicating the stability of (33). (Here, and below, the term "stability" is used loosely to indicate the absence of any tendency toward disastrous increase of irregularities in $\phi_{n, j}$ initiated by rounding errors and the like. It is hoped that, pending development of rigorous criteria, these imprecise indications of "stability in the mean" may provide useful hints to the working computer.)

A similar indication of stability can be displayed for the case,

$$|q\Delta t/\Delta x| \leqslant 1; \quad |q\Delta t/\Delta x| \leqslant 2p\sigma; \quad r \equiv 0. \tag{41}$$

This permits writing (33) in the form:

$$(1 + 2p\sigma)\psi_+ = (1 - 2p\sigma)\psi_- + q\Delta t/\Delta x \tag{42}$$

where

$$\psi_+ = (\phi_{n, j+1} - \bar{\phi}_{n, j})/(\phi_{n+1, j} - \phi_{n-1, j}); \quad \psi_- = (\phi_{n, j-1} - \bar{\phi}_{n, j})/(\phi_{n+1, j} - \phi_{n-1, j}). \tag{43}$$

Thus either

$$|\psi_+| \leqslant |\psi_-| \quad \text{or} \quad |\psi_+| \leqslant \tfrac{1}{2}. \tag{44}$$

This permits the inference that $|\phi_{n, j-1} - \phi_{n-1, j}| + |\phi_{n+1, j} - \phi_{n, j-1}|$ is (in the first event) decreased or (in the second event) at least not increased as a result of the replacement of $\phi_{n, j-1}$ by $\phi_{n, j+1}$. We can thus define a positive front-function

$$G = \sum_n |\phi_{n, j(n)} - \phi_{n-1, j(n-1)}| \tag{45}$$

which is non-increasing (except by reason of disturbance at the boundaries). The operation of the difference equation thus tends to bring the ϕ-distribution on the front to minimum total variation. This property gives a weaker indication of stability than that described above since G, unlike F, is not typically minimized by a unique ϕ-value. It nevertheless seems to justify describing the difference equation as stable.

The second restriction of (41) limits the usefulness of this argument if $2p\sigma$ becomes small. An argument which is then applicable may be made as follows: Define

$$\begin{aligned} M_L &= \phi_{n, j-1} - \phi_{n-1, j}; & M_R &= \phi_{n+1, j} - \phi_{n, j-1} \\ P_L &= \phi_{n, j+1} - \phi_{n-1, j}; & P_R &= \phi_{n+1, j} - \phi_{n, j+1}. \end{aligned} \tag{46}$$

Then equation (33), with $r \equiv 0$, may be shown to imply

$$(47)\quad (1 + q\Delta t/\Delta x)P_R^2 + (1 - q\Delta t/\Delta x)P_L^2 = (1 + q\Delta t/\Delta x)M_L^2 + (1 - q\Delta t/\Delta x)M_R^2 - 4p\sigma(1 + 2p\sigma)^{-2}\{M_L + M_R + (M_L - M_R)q\Delta t/\Delta x\}^2.$$

A front-function which displays a decreasing tendency may thus be written as

$$(48)\quad H = \sum_n (1 \pm q\Delta t/\Delta x)(\phi_{n,j(n)} - \phi_{n-1,j(n-1)})^2;$$

the $\pm$ as $j(n-1) = j(n) \pm 1$.

In (47) q is properly written q_{nj}. In (48) it becomes ambiguous, being either $q_{n-1,j(n)}$ or $q_{n,j(n)\pm1}$. Thus we can only assert that H is non-decreasing except by reason of variations in q with the advance of the front or disturbance introduced at the boundaries.

In equation (33) with nonvanishing r, the precise meaning of "stability" is not evident and no demonstration of properties approximating this concept are known to us. A preliminary qualitative examination failed to disclose any indication that variations in ϕ can grow to an alarming extent.

California Institute of Technology and Continental Oil Company — E. C. Du Fort

California Institute of Technology — S. P. Frankel

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[1] H. Lewy, "On the convergence of solutions of difference equations," *Studies and Essays Presented to R. Courant on his 60th Birthday*. New York, 1948, p. 211–214.

G. G. O'Brien, M. A. Hyman, & S. A. Kaplan, "A study of the numerical solution of partial differential equations," *Jn. Math. Phys.*, v. 29, 1951, p. 223–251.

F. John, "On Integration of Parabolic Equations by Difference Methods," *Comm. Pure Appl. Math.*, v. 5, 1952, p. 155–211.

R. P. Eddy, *Stability in the Numerical Solution of Initial Value Problems in Partial differential Equations*. Naval Ordnance Lab., *Memorandum* 10232, 1949.

[2] S. P. Frankel, "Convergence rates of iterative treatments of partial differential equations," *MTAC*, v. 4, 1950, p. 65–75.

[3] H. S. Carslaw & J. C. Jaeger, *Conduction of Heat in Solids*. Oxford, 1947, p. 280–281.

On Over and Under Relaxation in the Theory of the Cyclic Single Step Iteration

In order to speed up the sometimes very tedious computations in solving equations by the single step method, the device of the so-called "incomplete relaxation" (under or over relaxation) has been often used, although apparently no systematic discussion of this device has been as yet tried.[1]

It may seem, however, that in the case of the "relaxation procedure" the speeding up can be achieved in this way only in special cases, at least in the case of a symmetric positive definite matrix. Indeed, if the progress of the computation is measured by the decrease of a corresponding quadratic form $A(\zeta_k)$ depending on the k-th approximating vector ζ_k, we have the formula

$$(1)\quad A(\zeta_k) - A(\zeta_{k+1}) = q_k(2 - q_k)\,|r_{N_k}^{(k)}|^2/a_{N_kN_k}$$

where N_k is the index of the variable the value of which is improved at the k-th step, $r_{N_k}^{(k)}$ the corresponding "residual", q_k a coefficient characterizing the degree of the incomplete relaxation at this step. (We have usually $0 < q_k \leq 2$; for $0 < q_k < 1$, we have *under relaxation* and for $1 < q_k \leq 2$ *over relaxation*, while for $q_k = 1$ we have the usual complete relaxation.)[2] However, the above formula shows that *the decrease* of $A(\zeta_k)$ *is maximum* if q_k is taken as 1. Thus far the improvement in the case of $q_k \neq 1$ appears to be only possible if for such a value of q_k one of the residuals *at the next steps* will come out particularly large.

In what follows we discuss completely the case of a real 2×2 matrix, symmetric or not. It then turns out that, if the *cyclic* single step iteration converges, this convergence can indeed in all cases be essentially improved in using the incomplete relaxation with a convenient coefficient q, the same at each step. It turns out that even in many cases of unsymmetric matrices where the usual cyclic single step iteration diverges, it is made convergent in using a convenient incomplete relaxation.

The usual cyclic single step iteration is applied to a system of equations

$$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = y_\mu \qquad (\mu = 1, \cdots, n) \tag{2}$$

with a non-singular matrix $A = (a_{\mu\nu})$. We can assume without loss of generality that $y_\mu = 0$ $(\mu = 1, \cdots, n)$, since we can always obtain this by changing the origin. Then we get from one approximation vector $\xi = (x_1, \cdots, x_n)$ the next one $\xi' = (x_1', \cdots, x_n')$ by the formula

$$a_{\mu\mu} x_\mu' = a_{\mu\mu} x_\mu - \sum_{\nu=\mu}^{n} a_{\mu\nu} x_\nu - \sum_{\nu=1}^{\mu-1} a_{\mu\nu} x_\nu' \qquad (\mu = 1, \cdots, n). \tag{3}$$

This is replaced for the incomplete relaxation with a coefficient q by

$$a_{\mu\mu} x_\mu' = a_{\mu\mu} x_\mu - q \sum_{\nu=\mu}^{n} a_{\mu\nu} x_\nu - q \sum_{\nu=1}^{\mu-1} a_{\mu\nu} x_\nu' \qquad (\mu = 1, \cdots, n) \tag{4}$$

and this can be written in the form

$$a_{\mu\mu} x_\mu' + q \sum_{\nu=1}^{\mu-1} a_{\mu\nu} x_\nu' = a_{\mu\mu} (1 - q) x_\mu - q \sum_{\nu=\mu+1}^{n} a_{\mu\nu} x_\nu. \tag{5}$$

We now decompose A into the sum

$$A = L + D + R \tag{6}$$

where D is the diagonal matrix containing the main diagonal of A, while in L all elements on the diagonal and to the right of it and in R all elements on the diagonal and to the left of it vanish. We make here the assumption, which is usually made in the theory of the single step iteration, that none of the diagonal elements of A vanishes; then we can write (5) in the form

$$(D + qL)\xi' = ((1 - q)D - qR)\xi$$

and in solving this, since the triangular matrix $D + qL$ is non-singular, we obtain

$$\xi' = (D + qL)^{-1} ((1 - q)D - qR)\xi.$$

If now we put

$$(7) \qquad Q_q = (D + qL)^{-1} ((1 - q)D - qR)$$

we see that the k-th iterated vector ξ_k is obtained from the starting vector ξ_0 by the formula

$$\xi_k = Q_q^k \xi_0 \qquad (k = 1, 2, \cdots).$$

In order that our iteration be convergent for any starting vector ξ_0, it is necessary and sufficient that the maximum modulus Λ_q of the *characteristics* roots of the matrix Q_q be less than 1. The speed of the convergence is then measured by Λ_q. The smaller Λ_q, the faster is the convergence.

The characteristic equation of (7),

$$|\lambda E - (D + qL)^{-1} ((1 - q)D - qR)| = 0,$$

becomes, if the matrix of the left hand expression is multiplied by $D + qL$,

$$(8) \qquad |\lambda(D + qL) - ((1 - q)D - qR)| = 0.$$

We discuss this equation only in the case of a *real matrix A with $n = 2$.* Here the number

$$(9) \qquad u = (a_{12}a_{21})/(a_{11}a_{22})$$

is the characteristic constant of the problem. Our results are contained in the following four theorems.

THEOREM 1. *If $u > 1$ then for all q from $(0, 2)$ we have $\Lambda_q > 1$ and the process is divergent.*

THEOREM 2. *Suppose that $u < 1$ and put*

$$(10) \qquad q_0 = 2/(1 + (1 - u)^{1/2}).$$

Then with monotonically increasing q, Λ_q is monotonically decreasing for $q < q_0$ and monotonically increasing for $q > q_0$, so that the optimal value of Λ_q is

$$(11) \qquad \Lambda_{\text{opt}} = \Lambda_{q_0} = |u|/(1 + (1 - u)^{1/2})^2 \qquad (u < 1).$$

THEOREM 3. *Suppose that $u < 0$. Then a necessary and sufficient condition for the convergence of our procedure is that*

$$(12) \qquad 0 < q < q_1 = 2(1 + |u|^{1/2})^{-1}.$$

If $0 < u < 1$, we have convergence for all q with $0 < q < 2$.

THEOREM 4. *Suppose that $|u| < 1$. Then a necessary and sufficient condition for $\Lambda_q < \Lambda_1$ is that q be contained in the interior of the interval between 1 and $1 + u$.*

To prove these theorems we start from the corresponding case of (8)

$$\begin{vmatrix} (\lambda + q - 1)a_{11} & qa_{12} \\ \lambda q a_{21} & (\lambda + q - 1)a_{22} \end{vmatrix} = 0.$$

Without loss of generality we can assume that $a_{11} > 0$, $a_{22} > 0$. In dividing the first row and the first column by $a_{11}^{1/2}$ and the second row and second column by $a_{22}^{1/2}$ we can reduce A to the form

$$A = \begin{pmatrix} 1 & \beta \\ \alpha & 1 \end{pmatrix}, \qquad u = \alpha\beta.$$

Our equation for λ becomes now

$$(13)\quad N_q(\lambda) = \begin{vmatrix} \lambda + q - 1 & q\beta \\ \lambda q\alpha & \lambda + q - 1 \end{vmatrix} = \lambda^2 + \lambda(2q - 2 - q^2u) + (q - 1)^2 = 0.$$

For $q = 1$ we obtain

$$(14)\quad \Lambda_1 = |u|$$

and we see that the usual cyclic single step iteration converges, for an arbitrary starting vector, only if $|u| < 1$.

In the following discussion we assume $u \neq 0$. The discriminant Δ of the polynomial $N_q(\lambda)$ is

$$\Delta = \frac{1}{4} q^2u\ (uq^2 - 4\,q + 4).$$

This vanishes for $q = q_0 = 2/(1 + (1 - u)^{\frac{1}{2}})$, where obviously

$$(15)\quad \begin{cases} 1 < q_0 < 2 & (0 < u < 1) \\ 0 < q_0 < 1 & (u < 0), \end{cases}$$

while the other root $2/(1 - (1 - u)^{\frac{1}{2}})$ exceeds 2 if $0 < u < 1$ and is negative if $u < 0$. We see therefore that

$$(16)\quad \begin{cases} \Delta > 0 & (u > 1) \\ \operatorname{Sgn} \Delta = \operatorname{Sgn} u(q_0 - q) & (u < 1). \end{cases}$$

We consider now two cases according as $\Delta \leq 0$ or $\Delta > 0$. By (16), Δ is negative only if $u < 1$ and either $0 < u < 1$ and $q_0 < q \leq 2$ or $u < 0$ and $0 < q < q_0$. In both cases we have obviously from (13), $\Lambda_q = |q - 1|$ ($\Delta < 0$) and in particular

$$(17)\quad \begin{cases} \Lambda_q = q - 1 & (u > 0,\ \ 2 \geq q \geq q_0) \\ \Lambda_q = 1 - q & (u < 0,\ \ 0 < q \leq q_0). \end{cases}$$

In virtue of (16), the hypothesis $u > 1$ of theorem 1 is never realized for a negative Δ. Theorem 2 follows immediately from formulas (17). Theorem 3 follows from the fact that by (17) $\Lambda_q < 1$, while for $u < 0$, in any case $q_0 < q_1$ and q cannot exceed q_0 by (17). It follows finally from (17) that for $u > 0$ the condition

$$(18)\quad \Lambda_q < \Lambda_1 = |u|$$

is equivalent to $q < 1 + u$ while q remains $\geq q_0 > 1$. On the other hand, if we have $u < 0$, the condition (18) is equivalent to $q > 1 + u$, while q remains $\leq q_0 < 1$. Therefore theorem 4 and all our assertions are true in case $\Delta \leq 0$.

Now let $\Delta > 0$. If λ is the root of $N_q(\lambda)$ with $|\lambda| = \Lambda_q$, we have

$$(19)\quad \lambda = R + \epsilon\Delta^{\frac{1}{2}}, \quad R = \frac{u}{2}\left(q^2 - 2\,\frac{q - 1}{u}\right),$$

where $\epsilon = \operatorname{Sgn} R$. We have obviously

$$(20)\quad \Lambda_q = \epsilon\lambda.$$

The monotonicity of λ with respect to q depends on the sign of

$$\frac{d\lambda}{dq} = 2\frac{qu\lambda - \lambda - (q-1)}{2\lambda - q^2u - 2 + 2q}.$$

By (19), the denominator is equal to

$$2(\lambda - R) = 2\epsilon\Delta^{\frac{1}{2}}$$

and has the sign of ϵ. If we denote the numerator by δ, we have

(21) $$\delta = \lambda(qu - 1) + 1 - q$$

and therefore, by (20),

(22) $$\operatorname{Sgn}\frac{d\Lambda_q}{dq} = \operatorname{Sgn}\delta.$$

We deal first with the case $u > 1$. Here we have $\Delta > 0$, the roots of (13) are real and, since

$$N_q(1) = 1 - q^2u - 2 + 2q + q^2 - 2q + 1 = q^2(1-u) < 0,$$

one root λ exceeds 1, and so $\Lambda_q > 1$. We see that in this case we have divergence for any value of $q > 0$ and theorem 1 is proved.

Since for $u = 1$, A becomes singular, from now on we can make the assumptions

(23) $$u < 1, \quad q_0 \text{ is real}, \quad \Delta > 0.$$

We have then in particular from (16)

(24) $$\begin{cases} \text{either} & u > 0, \quad q < q_0, \quad q_0 > 1 \\ \text{or} & u < 0, \quad q > q_0, \quad q_0 < 1. \end{cases}$$

We prove now the following lemma, the proof of which is the main difficulty of the paper:

LEMMA. *Under the assumptions* (23) $u\delta$ *is always negative.*

Denote by δ_1 and δ_2 the two values of δ corresponding to the two roots of $N_q(\lambda)$. We have from (21) and (13) after some simplifications

(25) $$\delta_1 + \delta_2 = qu(q^2u - 3q + 2),$$

(26) $$\delta_1\delta_2 = (q-1)uq^2(1-u).$$

The expression on the right in (25) vanishes for $q = q_2$, where

(27) $$q_2 = 4/(3 + (9 - 8u)^{\frac{1}{2}}) < 1 \qquad (u < 1),$$

while the other root exceeds 2 for $u > 0$ and is negative for $u < 0$. Therefore we have

(28) $$\operatorname{Sgn}(\delta_1 + \delta_2) = \operatorname{Sgn} u(q_2 - q).$$

We will now consider separately the cases $u > 0$ and $u < 0$, and prove in the first case $\delta < 0$ and in the second $\delta > 0$.

In the case $u > 0$ we have $1 > u > 0$, and, since $\Delta > 0$, $q < q_0$. From (24) and (27) it follows that

(29) $$2 > q_0 > 1 > q_2.$$

If q exceeds 1, it follows from (29), (28), and (26) that

$$\delta_1 + \delta_2 < 0, \quad \delta_1\delta_2 > 0, \quad \delta_1 < 0, \quad \delta_2 < 0$$

and therefore $\delta < 0$.

If $q < 1$, it follows from (26) that $\delta_1\delta_2 < 0$. On the other hand the expression of R in (19), for $u > 0$, $q < 1$, becomes positive. We have therefore in this case $\epsilon = 1$, λ is the greater root of $N_q(\lambda)$ and since $qu - 1$ in (21) becomes negative, we have

$$\delta = \mathrm{Min}\,(\delta_1, \delta_2).$$

But then it follows from $\delta_1\delta_2 < 0$ that δ is negative in this case also.

We consider now the case $u < 0$. Since Δ is assumed positive, we have here by (16), $q > q_0$, while on the other hand from (24) and (27) it follows that

$$q_2 < q_0 < 1.$$

Since therefore in any case $q > q_2$, we have from (28)

$$\delta_1 + \delta_2 > 0. \tag{30}$$

On the other hand we conclude from (26)

$$\mathrm{Sgn}\,\delta_1\delta_2 = \mathrm{Sgn}(1 - q). \tag{31}$$

If therefore $q < 1$, we have $\delta_1\delta_2 > 0$ and from (30)

$$\delta_1 > 0, \quad \delta_2 > 0, \quad \delta > 0.$$

Suppose now $q > 1$; since in this case, by (19), R is negative, we have $\epsilon = -1$, $\Lambda_q = -\lambda$ and from (21) it follows now, since the coefficient of λ in (21) is negative, that

$$\delta = \mathrm{Max}(\delta_1, \delta_2).$$

But now we see from (30) that δ is again positive. Hence our lemma is proved.

We see now, from (22), that in the case of positive Δ, Λ_q monotonically decreases for $q < q_0$ and monotonically increases for $q > q_0$, as we already deduced from (17) in the case of negative Δ.

The minimum of Λ_q is obtained for $q = q_0$. We have, in applying for instance (17) and in using (10),

$$\begin{aligned} \Lambda_{q_0} &= |(1 - (1 - u)^{\frac{1}{2}})/(1 + (1 - u)^{\frac{1}{2}})| \\ &= |u|(1 + (1 - u)^{\frac{1}{2}})^{-2} \qquad (u < 1), \end{aligned} \tag{32}$$

and theorem 2 is completely proved.

The expression (32) is less than $|u|$ unless $u = 1$. For $|u| < 1$, we therefore always have an improvement in using incomplete relaxation with $q = q_0$, which is particularly pronounced for small $|u|$, since we have

$$\Lambda_{q_0} \sim |u|/4 \qquad (u \to 0). \tag{33}$$

As to the values of q for which we have convergence at all, we have, since $\Lambda_0 = 1$, convergence when q is in the interval $(0, q_0)$. On the other hand, the roots of (13) become, for $q = 2$,

$$2u - 1 \pm 2(u^2 - u)^{\frac{1}{2}}; \tag{34}$$

they are complex for $1 > u > 0$ and it follows from (17) that

$$\Lambda_2 = 1 \qquad (u > 0). \tag{35}$$

For $1 > u > 0$ we therefore have convergence for all q from $(0, 2)$.

If, on the contrary, $u < 0$, we obtain

$$\Lambda_2 = 1 - 2u + 2(u^2 - u)^{\frac{1}{2}} \qquad (u < 0)$$

and this exceeds 1. To obtain the value q_1 of q between q_0 and 2 for which Λ_q becomes 1, observe that

$$N_q(-1) = (2 - q)^2 + uq^2, \quad N_q(1) = q^2(1 - u).$$

For $u < 0$, $N_q(1)$ does not vanish while $N_q(-1)$ vanishes for

$$q_1 = 2/(1 + |u|^{\frac{1}{2}}) \qquad (u < 0). \tag{36}$$

Thus q_1 lies always between q_0 and 2, and since for $q = q_1$ the product of two roots of (13) is $(q_1 - 1)^2 < 1$, we have indeed

$$\Lambda_{q_1} = 1 \qquad (u < 0).$$

We have therefore for $u < 0$ convergence if and only if q lies in the interval $(0, q_1)$. In particular we have here for suitable values of q convergence for all negative u, but these values become small for large value of $-u$. Theorem 3 is thus completely proved.

It remains finally to answer the question for what values of q does the incomplete relaxation give any improvement at all, that is to say, $\Lambda_q < |u|$ (of course we assume here $|u| < 1$).

Now we have, by (14), $\Lambda_1 = |u|$ and the same is true for $q = 1 + u$:

$$\Lambda_1 = |u| = \Lambda_{1+u} \qquad (|u| < 1). \tag{37}$$

Indeed, we verify immediately that $1 + u \gtrless q_0$ according as u is positive or negative; therefore the roots of (13) are complex for $q = 1 + u$, and (37) follows from (17).

But now it follows from theorem 2 that we have improvement in the case of the incomplete relaxation (in contrast to the case $q = 1$) if and only if q lies in the interior of the interval between 1 and $1 + u$. Theorem 4 is thus proved.

We discuss finally a special example of a symmetric 2×2 matrix in which the improvement of the convergence due to the introduction of the incomplete relaxation is easily demonstrated explicitly.

Let
$$A = \begin{pmatrix} 1 & 3/5 \\ 3/5 & 1 \end{pmatrix}.$$

We have in this case by (10), (11), and (14)

$$\Lambda_1 = u = 9/25 > 1/3, \quad q_0 = 10/9, \quad \Lambda_{q_0} = 1/9. \tag{38}$$

The formulas for the cyclic single step iteration with $q = 1$ are in this case

$$x_1^{(\nu+1)} = -3x_2^{(\nu)}/5, \quad x_2^{(\nu+1)} = -3x_1^{(\nu+1)}/5 \tag{39}$$

and it is readily verified that, in putting $\xi_\nu = (x_1^{(\nu)}, x_2^{(\nu)})$, we have

$$\xi_\nu = (9/25)^\nu \, \xi_0.$$

In taking $\xi_0 = (1, 1)$ we have then

$$|\xi_\nu| = \sqrt{2}(9/25)^\nu. \tag{40}$$

Consider on the other hand the over relaxation with the value of $q = 10/9$. Here the components of the approximating vectors are to be computed from the equations

$$x_1^{(\nu+1)} = -x_1^{(\nu)}/9 - 2x_2^{(\nu)}/3, \quad x_2^{(\nu+1)} = -x_2^{(\nu)}/9 - 2x_1^{(\nu+1)}/3.$$

If we put $\xi_\nu = (x_1^{(\nu)}, x_2^{(\nu)})$ and assume again $\xi_0 = (1, 1) = \xi_0$, we obtain as is readily verified

$$\xi_\nu = 3^{-2\nu-1}(3 - 24\nu, 3 + 8\nu) \qquad (\nu = 0, 1, \cdots).$$

Here we have

$$|\xi_\nu| \sim 8(10)^{\frac{1}{2}}\nu 9^{-\nu}/3 \qquad (\nu \to \infty).$$

We give in what follows a table of the initial values of $|\xi_\nu|$ and $|\xi_\nu|$.

ν	$\lvert\xi_\nu\rvert$	$\lvert\xi_\nu\rvert$
1	0.5091	0.8780
2	0.1833	0.2010
3	0.06598	0.03388
4	0.02375	0.005048
5	0.008551	0.0007037

Although the difference between q_0 and 1 is very small, in fact 1/9, the improvement is already observed at ξ_3 and becomes more and more pronounced from there on.

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[1] S. P. FRANKEL, "Convergence rates of iterative treatments of partial differential equations," *MTAC*, v. 4, 1950, p. 65–75.
[2] A. M. OSTROWSKI, *On the Linear Iteration Procedures for Symmetric Matrices*. NBS *Report* no. 1844, August 1952, p. 23, (68).

The Accuracy of Numerical Solutions of Ordinary Differential Equations

1. Introduction. The present paper describes a general method by which the random and systematic errors may be estimated of numerical solutions of any systems of ordinary differential equations. The errors arise from the accumulation of rounding-off errors, and from the use of erroneous formulas for performing the numerical integrations. The estimation is based on the properties of the solutions of the system of equations adjoint to the variational equations of the problem, and is applicable to any method of integration.

The present method was described by the author at a meeting of the American Astronomical Society on February 1, 1946 at Columbia University. Development of the method resulted from a conversation with CHARLES B. MORREY, JR., who explained to the author properties of the solutions of adjoint equations with which the author was not then familiar. The procedure seems intuitively obvious and straightforward, and it has not been published earlier both for this reason and because it was understood that HANS RADEMACHER was planning to publish a similar and possibly independent treatment. It now appears, however, that Rademacher[1] was concerned with the accuracy of particular methods of integration, while the present method is applicable to any integration procedure that may be employed. There are still other methods for estimating the accuracy of numerical solutions of special types of ordinary differential equations. For example, BROUWER[2] has made special studies of the accuracy of numerical integrations, by the Crommelin-Cowell method, of the orbital differential equations of dynamical astronomy.

A virtue of the present method is its generality; but there are alternative general methods, possibly just as good, which may not have been published. The author understands from conversations with L. H. THOMAS, that he has made use of general procedures not involving the adjoint equations. The main justification for publishing a description of the author's procedure is his hope that it may help others to select computational procedures, for numerical integrations, that will yield results of desired accuracy.

2. The Adjoint Equations. Consider the system of n first-order differential equations

$$\dot{x}_i = \sum_{j=1}^{n} a_{i,j}x_j + b_i; \quad i,j = 1,2,\cdots,n \tag{1}$$

where the n^2 quantities $a_{i,j}$ and the n quantities b_i may vary with the independent variable, t. Let λ_i be a set of variables satisfying the adjoint system of equations

$$-\dot{\lambda}_i = \sum_{j=1}^{n} a_{j,i}\lambda_j. \tag{2}$$

Since

$$\begin{aligned} (d/dt)\sum_{i=1}^{n} x_i\lambda_i &= \sum_{1}^{n} \dot{x}_i\lambda_i + \sum_{1}^{n} x_i\dot{\lambda}_i \\ &= \sum_{i,j} a_{i,j}x_j\lambda_i - \sum_{i,j} a_{j,i}x_i\lambda_j + \sum_i b_i\lambda_i \\ &= \sum_i b_i\lambda_i \end{aligned} \tag{3}$$

it follows that

$$\sum_{i=1}^{n} x_i\,(A)\,\lambda_i\,(A) = \sum_i x_i\,(0)\,\lambda_i\,(0) + \int_0^A \sum_i b_i\lambda_i\,dt. \tag{4}$$

3. Application. From any system of ordinary differential equations there can be derived a set of variational equations of the form (1) where the $x_i(t)$'s

are differences, between the exact solution of the original equations corresponding to the desired initial conditions, and any neighboring exact solution not subject to the desired initial conditions. If a single error were made in the course of solving the original system of equations by a scheme of stepwise integration that was otherwise perfect, the solution would be exact before the error was made; for later values of the independent variable the solution would still be an exact solution of the original differential equations but would correspond to altered initial conditions. The x_i's would all be zero before the error, and would grow after it in accordance with the equations of the form (1), with b_i's that were zero for all steps except the one in which the error was made.

If, because of defective methods of calculation or for any other reason errors $\epsilon_i(t)$ are introduced into the i'th variable x_i at a particular step ($t - w$ to t, say) in the solution of the original system of differential equations, one may consider the ϵ's to have been introduced by b_i's in (1) such that

$$\epsilon_i(t) = \int_{t-w}^{t} b_i(t)\, dt$$

and that are zero outside of the interval $(t - w, t)$. One may consider approximately that

$$b_i(t) = (1/w)\epsilon_i(t)$$

throughout the interval and therefore approximately that

$$\int_{t-w}^{t} b_i(t)\, \lambda_i(t)\, dt = \epsilon_i(t)\, \lambda_i(t).$$

Thus by (4) the resulting final error (at $t = A$, say) in a particular variable (e.g., x_1 say) is

$$x_1(A) = \sum_{i=1}^{n} \epsilon_i(t)\, \lambda_i(t)$$

and if errors are introduced at all steps

$$x_1(A) = \sum_{\text{all steps}} \sum_{i=1}^{n} \epsilon_i(t)\, \lambda_i(t) \tag{5}$$

provided that the λ's are any solution of (2) satisfying the boundary conditions

$$\lambda_1(A) = 1; \quad \lambda_j(A) = 0, \quad j \neq 1.$$

4. Truncation and Rounding Errors. Equations (4) and (5) provide means for predicting the errors of numerical solutions of systems of ordinary differential equations. Rough solutions of the equations (2), based on a rough solution of the original equations, are in practice adequate. Rounding-off errors of the usual hand-made variety are drawn from populations whose means are zero, and whose individuals are uniformly distributed from $-1/2$ to $1/2$ in units of the last digit. Their variance is thus $1/12$ in such units. The resulting variance of a final value like $x_1(A)$ is given by

$$\sigma^2_{x_1(A)} = \sigma_1^2 \sum \lambda_1^2(t) + \sigma_2^2 \sum \lambda_2^2(t) + \cdots + \sigma_n^2 \sum \lambda_n^2(t) \tag{6}$$

where σ_i^2 is the variance of the rounding-off errors introduced into the variable x_i at any step, and where the sums are taken over all steps. The preceding equation is general; if the rounding-off errors are not of the usual hand-made sort it is still valid. If the rounding-off errors do not come from populations with zero means, then a bias, or systematic error is introduced whose final value has the population mean

$$\overline{x_1}\,(A) = \sum_t \sum_i M_i(t)\, \lambda_i(t)$$

where $M_i(t)$ is the population mean of the error $\epsilon_i(t)$, conceivably a function of t.

Besides rounding-off errors, "truncation" errors are introduced by the circumstance that the formulas employed for integrations are erroneous. Whatever the formulas are, and however they are employed, iterated or not, any particular method of integration applied to a particular system of differential equations always corresponds to the exact solution of a system of difference equations rather than of the original differential equations. The particular method of integration thus corresponds to an exact solution of a system of differential equations somewhat different from the original differential equations. It is always possible to evaluate, approximately, the differences between the original differential equations and those that the scheme is exactly solving, then to find the appropriate b_i's in the variational equations of the form (1), and finally to apply (4) to predict the final errors, thus

$$x_1(A) = \int_0^A \sum_i b_i(t)\, \lambda_i(t)\, dt \tag{7}$$

in which, as usual, the λ's must be chosen to satisfy the boundary conditions

$$\lambda_1(A) = 1; \quad \lambda_j(A) = 0, \quad j \neq 1.$$

Alternatively, one can find appropriate truncation errors $\epsilon_i(t)$ and then apply equation (5).

For planning purposes, no great accuracy in the calculations of accuracy is necessary, and no great accuracy should be sought. Even rough calculations are expected to suffice to decide how many digits, what size of steps, and what scheme of integration to employ.

5. Example. An example, suggested by WERNER LEUTERT, will be given of the use of the preceding method by an application to the non-linear differential equation of the first order

$$\dot{y} = (3/2)\, t\, y^{-1/3} \tag{8}$$

which can be integrated analytically but which will be treated as though it can not be. Suppose that one wishes to integrate this equation numerically, starting with the value

$$y(1) = 1,$$

as far as $y(5)$; and that one wishes the value $y(5)$ to be accurate "to the third place" of decimals. One wishes to use the scheme of integration defined

by the approximate formula

$$\nabla y = \left[1 - \frac{\nabla}{2} - \frac{\nabla^2}{12}\right] w\dot{y} \tag{9}$$

in which the differences are backward, or ascending, and in which w is the length of a step. One wishes to determine w, and the number of decimals to retain in the calculations.

The variational equation corresponding to (1) is

$$\dot{x} = -\frac{1}{2} t\, y^{-4/3}\, x \tag{10}$$

and the adjoint equation corresponding to (2) is thus

$$\dot{\lambda} = \frac{1}{2} t\, y^{-4/3}\, \lambda. \tag{11}$$

A rough integration of (8) must first be accomplished, and then a rough integration of (11) to find λ. Such integrations have been accomplished by the aid of a ten-inch slide rule, and the results appear in the first four columns of the following table:

t	y	2.27λ	λ	$\lambda D^4 y$
1	1	1	.44	.25
2	2.80	1.44	.63	.06
3	5.15	1.76	.78	.03
4	7.95	2.03	.89	.01
5	11.15	2.27	1.00	.01

No attempt has been made to obtain results correct to the second place of decimals, although two places were retained. The integration for λ with the starting value unity led to a value 2.27 at $t = 5$; the fourth column contains λ adjusted to have the value unity at $t = 5$.

Consideration of the difference formula (9) shows that it corresponds substantially to the differential equation

$$Dy - (1/24)\, w^3 D^4 y + (11/720)\, w^4 D^5 y + \cdots = (3/2)\, t\, y^{-1/3} \tag{8'}$$

instead of to equation (8), so that approximately

$$b(t) = (1/24)\, w^3 D^4 y$$

or

$$\epsilon(t) = (1/24)\, w^4 D^4 y;$$

these results could have been obtained directly from the term of lowest order that has been omitted from the right-hand member of equation (9). By equation (5) or equation (7) the truncation error at $t = 5$ is

$$x(5) = (w^3/24) \int_1^5 D^4 y(t)\, \lambda(t)\, dt.$$

Values of D^4y are obtained from equation (8); values of $D^4y(t)\ \lambda(t)$ are tabulated above; a quadrature furnishes the result

$$x(5) = w^3/120. \tag{12}$$

It is noticed that a unit error in $w\dot{y}$ at any stage introduces a unit error in y when the scheme of integration is (9). Hence by equation (6) the variance of $y(5)$ arising from rounding errors is

$$\begin{aligned}\sigma^2_{y(5)} &= (1/12) \sum_{t=1}^{5} \lambda^2(t)\\ &= (1/12\,w) \int_1^5 \lambda^2(t)\,dt\\ &= 1/5\,w\end{aligned}$$

by quadratures, very nearly, in units corresponding to the last place retained in $w\dot{y}$.

To obtain a value of $y(5)$ accurate "to the third place" of decimals, one equates the right-hand member of equation (12) to 0.0005 and solves for w. It is found that w is .39. One can adopt a value $w = .4$, if one will tolerate a bias error of 0.00053 in $y(5)$; this is tolerated, and the value $w = .4$ is adopted. A number fairly rounded to the nearest 0.001 has a rounding error whose variance is 1/12 in units of the sixth place. It is therefore reasonable to require that the variance of $y(5)$ from the accumulation of rounding errors should be smaller than 1/12 in the sixth place. If only three decimals were retained in the values of $w\dot{y}$ then the variance of $y(5)$ would be $1/5w$ or 1/2 in the sixth place, which is too large to be acceptable. With four decimals, the variance is 1/2 in the eighth place, or 1/200 in the sixth, which is better than is needed. Therefore four places of decimals should be retained in values of $w\dot{y}$.

The definitive integration of equation (8) by the scheme (9) was next accomplished, with steps of 0.4 and with four places of decimals. The value obtained was

$$y(5) = 11.1810.$$

The correct value of $y(5)$, obtained by analytical solution of (8), is

$$\begin{aligned}y(5) &= 5^{3/2}\\ &= 11.18034\cdots\cdots\end{aligned}$$

showing that the error of $y(5)$ obtained by the numerical integration is 0.00066, and that $y(5)$ is substantially as accurate as was desired and as was predicted.

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[1] See, for instance, HANS RADEMACHER, "On the accumulation of errors in numerical integration on the Eniac," lecture 19 of *Theory and Techniques for Design of Electronic Digital Computers*, v. 2, Moore School of Electrical Engineering, University of Penn., 1947.

[2] DIRK BROUWER, "On the accumulation of errors in numerical integration," *Astronomical Jn.*, v. 46, p. 149, 1937.

The Zeros of the Partial Sums of e^z

The location of the zeros of certain entire functions has received considerable attention in the literature, and it was suggested by R. S. VARGA that a table of the zeros of certain truncated power series might be of interest. Accordingly, the zeros of the truncated exponential series $S_n(z) = \sum_0^n \frac{z^k}{k!}$ have been computed for values of n up to 23. They appear in the accompanying table with the complex zeros ordered as to modulus and with the real zero printed last.

Table of Zeros of $S_n(z)$

$n = 2$

−1.0000 0000 0000 ±	1.0000 0000 0000i

$n = 3$

−0.7019 6418 1008 ±	1.8073 3949 4452i
−1.5960 7163 7983	

$n = 4$

−1.7294 4423 1067 ±	0.8889 7437 6121i
−0.2705 5576 8932 ±	2.5047 7590 4362i

$n = 5$

−1.6495 0283 1735 ±	1.6939 3340 4349i
0.2398 0639 3753 ±	3.1283 3502 5970i
−2.1806 0712 4035	

$n = 6$

−2.3618 1018 0482 ±	0.8383 5027 7917i
−1.4418 0139 0549 ±	2.4345 2268 1808i
0.8036 1157 1031 ±	3.6977 0175 3629i

$n = 7$

−2.3798 8388 3168 ±	1.6289 9897 6372i
−1.1472 0068 9937 ±	3.1240 3923 8058i
1.4065 8592 8087 ±	4.2250 6684 4949i
−2.7590 0270 9962	

$n = 8$

−2.9645 9950 5160 ±	0.8088 7832 7313i
−2.2864 2928 4171 ±	2.3777 1166 7793i
−0.7887 9362 0387 ±	3.7718 1078 3950i
2.0398 2240 9719 ±	4.7186 1488 3923i

$n = 9$

−3.0386 4807 2936 ±	1.5868 0119 5758i
−2.1108 3980 0302 ±	3.0899 1092 8725i
−0.3810 6984 5663 ±	4.3846 4453 3145i
2.6973 3346 1536 ±	5.1841 6206 2649i
−3.3335 5148 5269	

$n = 10$

−3.5538 7599 3928 ±	0.7894 2208 2895i
−3.0155 3577 0425 ±	2.3352 2385 7750i
−1.8716 6001 0419 ±	3.7701 9023 1409i
0.0662 0154 6301 ±	4.9676 7937 0404i
3.3748 7022 8472 ±	5.6260 2017 9698i

$n = 11$

−3.6641 5160 2429 ±	1.5570 4385 7798i
−2.9170 5082 0338 ±	3.0563 7182 9831i
−1.5814 4081 0406 ±	4.4223 5472 7709i
0.5460 7815 8920 ±	5.5249 0494 6051i
4.0692 9095 3063 ±	6.0474 9237 9889i
−3.9054 5175 7616	

$n = 12$

−4.1356 0823 9264 ±	0.7755 4204 9110i
−3.6888 9710 2446 ±	2.3027 5714 9247i
−2.7579 8923 1191 ±	3.7525 4838 2353i
−1.2491 2514 3358 ±	5.0495 5510 7284i
1.0534 2363 9656 ±	6.0594 9143 3864i
4.7781 9607 6918 ±	6.4511 7633 7446i

$n = 13$

−4.2712 4352 2040 ±	1.5348 5553 2416i
−3.6448 0690 0313 ±	3.0273 4405 4885i
−2.5489 2149 0908 ±	4.4259 0791 2207i
−0.8813 4146 1503 ±	5.6544 1733 8744i
1.5843 1494 9756 ±	6.5740 0727 9114i
5.4997 0440 2346 ±	6.8391 5923 4366i
−4.4754 1195 4676	

$n = 14$

−4.7125 8682 7652 ±	0.7651 0266 4661i
−4.3307 0823 3773 ±	2.2772 3593 3932i
−3.5438 3446 4412 ±	3.7319 2259 2644i
−2.2976 9841 1869 ±	5.0783 9242 5805i
−0.4831 5856 3940 ±	6.2391 4928 3405i
2.1356 7746 7731 ±	7.0705 6793 5765i
6.2323 0903 3917 ±	7.2131 4836 1604i

$n = 15$

−4.8669 6227 5703 ±	1.5176 2913 7005i
−4.3271 6518 2166 ±	3.0027 7099 6726i
−3.3948 7438 4748 ±	4.4177 0410 7602i
−2.0103 3299 7335 ±	5.7117 2749 1737i
−0.0585 5212 5116 ±	6.8056 2430 6791i
2.7050 4956 8386 ±	7.5509 4048 4655i
6.9747 8093 2988 ±	7.5745 6159 4666i
−5.0438 8707 2612	

Table of Zeros of $S_n(z)$—*Continued*

$n = 16$

−5.2863 1780 3783	±	0.7569 4362 0482i
−4.9527 8138 2083	±	2.2566 6238 4481i
−4.2704 2428 9874	±	3.7119 3355 5596i
−3.2047 4966 4526	±	5.0858 8483 8575i
−1.6915 4717 7729	±	6.3274 3910 8059i
0.3892 9335 7090	±	7.3554 4605 9723i
3.2904 2476 4254	±	8.0166 1911 0269i
7.7261 0219 6653	±	7.9245 9187 5073i

$n = 17$

−5.4551 0328 9602	±	1.5038 3976 7334i
−4.9806 7273 2968	±	2.9819 4653 1042i
−4.1680 2106 9422	±	4.4053 7601 5639i
−2.9788 2514 8596	±	5.7375 9875 5455i
−1.3451 2663 7403	±	6.9268 7649 6075i
0.8577 8157 7290	±	7.8899 9871 2830i
3.8901 4238 0621	±	8.4688 8076 9632i
8.4854 1859 0153	±	8.2642 5429 2176i
−5.6111 8734 0141		

$n = 18$

−5.8576 9828 4668	±	0.7503 7782 8924i
−5.5616 1575 6798	±	2.2397 2181 6511i
−4.9588 1020 0174	±	3.6935 9122 5275i
−4.0258 8765 2371	±	5.0838 2295 5316i
−2.7214 0644 5610	±	6.3738 9942 0060i
−0.9741 5991 2999	±	7.5112 3498 4979i
1.3447 6355 9230	±	8.4104 8615 4221i
4.5028 0973 4285	±	8.9088 2705 4271i
9.2520 0495 9109	±	8.5944 2118 7876i

$n = 19$

−6.0379 0688 9211	±	1.4925 3401 9264i
−5.6146 0932 9746	±	2.9641 6550 4574i
−4.8936 3517 6394	±	4.3919 0787 8943i
−3.8487 8978 9420	±	5.7480 1406 1058i
−2.4360 0924 7000	±	6.9957 5579 1547i
−0.5812 0465 9773	±	8.0815 7686 4150i
1.8484 3722 1795	±	8.9179 6283 4128i
5.1272 4541 2251	±	9.3374 1621 1924i
10.0252 3949 6630	±	8.9158 4881 4122i
−6.1775 3407 8278	±	

$n = 20$

−6.4273 0264 0861	±	0.7449 7139 4210i
−6.1611 0097 0908	±	2.2255 2308 7417i
−5.6209 8880 2007	±	3.6770 7242 0762i
−4.7903 2735 9920	±	5.0773 1588 8906i
−3.6406 2228 5764	±	6.3986 7391 1969i
−2.1255 4693 1251	±	7.6040 5463 9680i
−0.1684 0640 7437	±	8.6388 4961 0732i
2.3672 7408 2252	±	9.4133 5826 0670i
5.7624 3706 9983	±	9.7554 8801 5501i
10.8045 8424 5908	±	9.2291 9790 4677i

$n = 21$

−6.6167 8934 1171	±	1.4830 8180 5631i
−6.2346 1169 3725	±	2.9488 4393 5502i
−5.5863 1103 9618	±	4.3785 4166 8723i
−4.6531 2333 7897	±	5.7502 8769 4696i
−3.4046 1507 1622	±	7.0364 9649 6532i
−1.7924 6476 9782	±	8.1996 0590 2952i
0.2624 1464 0116	±	9.1839 0151 8883i
2.8999 6466 8596	±	9.8974 9663 3020i
6.4075 0945 9944	±	10.1637 8334 0590i
11.5895 7008 7260	±	9.5350 4977 9337i
−6.7430 9089 3669		

$n = 22$

−6.9955 1940 2365	±	0.7404 3625 9274i
−6.7537 1576 2515	±	2.2134 4348 1101i
−6.2643 4318 6033	±	3.6622 7895 1044i
−5.5151 2445 5329	±	5.0687 9151 8941i
−4.4855 3901 9504	±	6.4113 1616 4046i
−3.1434 8080 8797	±	7.6621 3795 8098i
−1.4388 3815 7314	±	8.7831 4914 1824i
0.7096 9887 5451	±	9.7174 9498 5677i
3.4453 7712 5688	±	10.3711 1273 0512i
7.0616 9981 9937	±	10.5629 5976 6282i
12.3797 8501 1070	±	9.8339 1911 3179i

$n = 23$

−7.1926 8590 7451	±	1.4750 5919 5082i
−6.8443 1411 8665	±	2.9355 2075 2798i
−6.2551 3130 4907	±	4.3657 9349 6476i
−5.4112 0644 7035	±	5.7481 1045 3530i
−4.2905 2911 0822	±	7.0608 9517 9930i
−2.8595 2302 8914	±	8.2762 1397 0162i
−1.0664 4595 7935	±	9.3553 6015 8909i
1.1720 9815 7227	±	10.2403 1777 0398i
4.0025 2534 8326	±	10.8348 6485 1671i
7.7243 3865 4741	±	10.9536 0414 4523i
13.1748 6483 8907	±	10.1262 6417 8727i
−7.3079 8221 4646		

The work was done on the Harvard Mark IV Calculator[1] using the quadratic factor method[2] together with a routine which supplied initial approximations chosen on a rectangular mesh in the complex plane. Mark IV operates with a fixed decimal point and carries sixteen decimal digits, but to reduce round-off errors a floating point routine was used. As a final check, the sum and product of the roots of each polynomial were compared with the appropriate coefficients of $S_n(z)$, and agreement to twelve significant digits was obtained for all $n < 21$.

The plot of the zeros in Fig. 1 exhibits the regularity of the family of curves joining the n zeros of a given $S_n(z)$. The broken curves which join zeros of the same rank (when ordered as to modulus for each n) have immediate application in that the zeros of the partial sum $S_n(z)$ may be located approximately from a knowledge of the zeros of the partial sums of lower order. This could be used to provide good first approximations in extending the table.

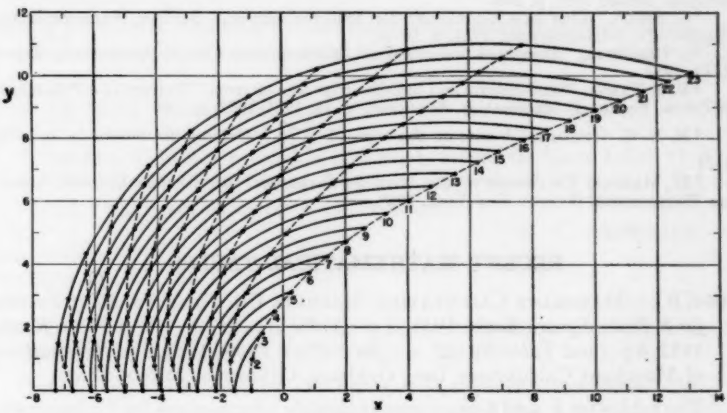

FIG. 1.

It is interesting to consider the present numerical results in the light of the following known properties of $S_n(z)$:

(1) $\lim\limits_{n\to\infty} \dfrac{r_n}{n} = \dfrac{1}{2} - \dfrac{1}{e\pi}$ where r_n is the number of zeros of $S_n(z)$ lying in the right half-plane.[3]

(2) $S_{2n}(z)$ and $S_{2n+1}(z)$ each have $2n$ complex zeros.[4]

(3) The semi-infinite strip[5] $|y| < \sqrt{6}$, $x > 0$, contains no zeros of any $S_n(z)$. Figure 1 suggests that much larger zero-free regions exist.

(4) Every zero of $S_n(z)$ satisfies the inequality[6]

$$n > |z| > \frac{n}{e^2}.$$

(5) The zero of smallest modulus of $S_{2n+1}(z)$ is real and negative. An unpublished proof has been given by D. J. NEWMAN.

(6) The convex polygon formed by the zeros of $S_n(z)$ encloses all the zeros of the partial sums of lower order. This is based on the fact that $S_n'(z) = S_{n-1}(z)$ and on the following theorem:[7] Any convex

polygon which contains all the zeros of a polynomial $p(z)$ also contains all the zeros of the derivative $p'(z)$.

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[1] *A Description of the Mark IV Calculator*. Computation Laboratory of Harvard University, *Annals*, v. 28 (In preparation).
[2] W. E. Milne, *Numerical Calculus*. Princeton 1949, p. 53; or D. R. Hartree, *Numerical Analysis*. Oxford 1952, p. 205.
[3] G. Szegö, "Über eine Eigenschaft der Exponentialreihe," Berliner Mathematischen Gesellschaft, *Sitzungsberichte*, 1924, p. 50–64.
[4] J. Berghuis, "Truncated power series," Mathematical Centre, Amsterdam, *Report* R173, 1952.
[5] R. S. Varga, "Semi-infinite and infinite strips free of zeros," Universita e Politecnico di Torino, Seminario Matematico, *Rendiconti*, v. 11, 1951–1952, p. 289.
[6] K. S. K. Iyengar, "A note on the zeros of $\sum_0^n \frac{x^r}{r!}$." *Mathematics Student*, v. 6, 1938, p. 77.
[7] M. Marden, *The Geometry of the Zeros of a Polynomial in a Complex Variable*. American Mathematical Society. New York, 1949.

RECENT MATHEMATICAL TABLES

1094[B].—Marchant Calculating Machine Co. *Table No. 80 of Factors for 8-Place Square Roots*, 1951, 4 p.; *Table No. 81* ··· *for 6-Place Roots*, 1952, 4 p.; and *Table No. 82* ··· *for 5-Place Roots*, 1952, 2 p. *Publications* of Marchant Calculators, Inc., Oakland, California. 21.5 × 28 cm.

The tables for 5- and 6-place roots resemble a former one for 5-place roots of the same publisher (*MTAC*, v. 1, p. 356). As in the former table the root results from adding a tabular number to the number N of which root is desired, and dividing this sum by an adjacent tabular number. The table for 8-place roots requires two divisions but without need of intermediate copying. Mathematically the table for 8-place roots is equivalent to a similar two-division table reported in *MTAC*, v. 5, p. 180.

Doubtless the most welcome of these tables will be the one for 6-place roots because this number of places is specified by the majority of work sheets for general surveying and military uses. There has not been available a means of obtaining 6-place roots by the use of tables of this type and a single division.

Significant improvement has been made in tables Nos. 81 and 82 as compared with the previous 5-place table (1) by eliminating need of determining which of two arguments is nearer to N, (2) by reducing the number of tabulated divisors—resulting from using a single column of divisors for the entire range of N from 1 to 100 and from altering the range of error from (0 to +5) to (−5 to +5) in units of last place, (3) by using the leeway afforded by integer arguments to reduce the number of digits of the divisors to as few as possible.

The method of selecting argument intervals and the error expression of tables of the single-division type are described in Willers, *Practical Analysis*, Dover edition, Art. 6–21, 1948 (*MTAC*, v. 3, p. 493). Obvious alteration of the error expression is necessary when applying them to these tables because of change of the range of error.

The tables are suitable for use with any of the usual types of desk calculating machines, though the manipulative description accompanying table 80 refers to the Marchant machine. The tables were prepared under the direction of H. T. AVERY who asks that aid rendered by H. J. REYNOLDS be acknowledged.

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1095[B].—VICTOR THÉBAULT, *Les Récréations Mathématiques* (*Parmi les nombres curieux*). Paris, Gauthier-Villars, 1952, vi, 299 p., 15.5 × 23.5 cm.

P. 12: Table of the 87 squares formed by the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 ,9, each once and only once.
P. 89–119: Tables of squares of integers 1(1)1000 for bases 2(1)9, 11, 12.
P. 226–230: The tables of squares with (a) 3, (b) 5, (c) 6, (d) 7, (e) 8, (f) 9 digits occurring once and only once.

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1096[C,K].—M. S. BARTLETT, "The statistical significance of odd bits of information," *Biometrika*, v. 39, 1952, p. 228–237.

Table I (p. 230) lists to 4D for $p = 0(.01)1$ the functions $-\ln p$; $-p\ln p$; $p\ln^2 p$; $-p\ln p - (1-p)\ln(1-p)$; and $p(1-p)\ln^2(p/(1-p))$. An example is given using this table to compute an "information" function.

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1097[C].—ALEXANDER JOHN THOMPSON, *Logarithmetica Britannica, being a Standard Table of Logarithms to Twenty Decimal Places. Part II, Numbers 20,000 to 30,000 together with General Introduction*. Cambridge University Press, 1952. vi, [102], xcviii, [xi] p., 21.6 × 27.7 cm., 45 shillings; New York price $8.50.

This is the ninth and final part of the very notable work which Dr. Thompson started 30 years ago, as a publication for KARL PEARSON'S *Tracts for Computers*, Department of Statistics of the University of London, to commemorate the tercentenary of HENRY BRIGGS' publication of *Arithmetica Logarithmica*, 1624. This work of Briggs (1561–1631) contained log N to 14D for $N = 1(1)20000$, $90000(1)101000$, and the square roots of the integers [1(1)200; 11D], with first differences in each case; (see *MTAC*, v. 1, p. 170; v. 2, p., 94,196). On pages lxxviii–lxxxiii of this part II is a list of errors of more than a unit in the values of log N in this Briggsian table. Briggs had practically completed the computation of log N for $N = 20001(1)90000$ when ADRIAN VLACQ published *Arithmetica Logarithmica* (1628) for log N, $N =$ [1(1)100000; 10D], and characterized his volume as the second edition of the work by Briggs. The completion by Vlacq of a 10D table from the earlier 14D tables of Briggs was a comparatively simple matter. But his

action in rushing into print with such a publication can scarcely be termed other than reprehensible.

The first part of Dr. Thompson's work, $N = 90{,}000\text{–}100{,}000$, appeared in 1924, and this was followed by other parts in 1927, 1928, 1931, 1933, 1934, 1935, 1937. Thus 15 years elapsed after the appearance of 8 parts, before the final part, completing the table for $N = 10000(1)100000$, was published. For brief reviews of parts VI and VII see *MTAC*, RMT **23**, **42**, **65**, v. 1, p. 4, 5, 7, that is, *Scripta Mathematica*, v. 2, 1934, p., 196; v. 3, 1935, p., 192; v. 4, 1936, p., 201.

Since 100 logarithms are displayed on each page, there are in the whole work 900 pages occupied by the fundamental table. To introduce this table there are in the present part 108 pages of preliminary matter. In other parts there was further preliminary matter to which we shall presently refer. Hence in this part are the necessary title-pages and indices for binding the whole work into two volumes. Of the useful factoring Table F (logarithms of $1 + N/10^7$, $1 + N/10^{10}$, and $1 + N/10^{13}$, $N = [0(1)1000; 21D]$) there are two copies, one for each volume. Reprints of the complete work in two bound volumes are now available at the publisher. For completing sets, copies of the first eight published parts are available at 21 shillings each.

In part I (1934) is a two-page facsimile of the original will of HENRY BRIGGS, the first page in his own handwriting; part III (1937) has four pages with facsimiles of five letters of Briggs, and a Note by Dr. THOMPSON; in part IV (1928) there is a facsimile of the title-page of the work in which Briggs's Treatise on the Northwest Passage to the South Sea was published in 1622; two pages of errors in *Arithmetica Logarithmica*, 1624, are also given; part V (1931) has a facsimile of the title-page of *Arithmetica Logarithmica*; and part VI (1933) has a facsimile of a Briggs' letter to JOHN PELL.

The *Mirifici Logarithmorum Canonis Descriptio* was first published in 1614 by JOHN NAPIER (1550–1617). The second edition in 1619, prepared after Napier's death by his son ROBERT, and Briggs, contained an edited earlier work of Napier: *Mirifici Logarithmorum Constructio, Canonis* in which Napier called logarithms "artificial numbers." See my article, "Napier's *Descriptio and Constructio*," Amer. Math. Soc., *Bull.*, v. 22, 1916, p. 182–187. Part VII (1935) has four pages (three of 6 facsimile pages) illustrating the relation of Henry Briggs to Napier's *Constructio Canonis* with a Note by KARL PEARSON. It is by no means clear how Pearson reasons that the word "logarithm" used by Napier in his work of 1614 "must have been added by Briggs or Robert Napier," because Napier had used a different term in his *Constructio*.

Parts VIII (1927) and IX (1924) contained a number of pages of introductory matter by Thompson and in the latter a "Prefatory Note" by Karl Pearson, and also a facsimile of the title and specimen pages of the excessively rare 1617 tract (16 p.): *Logarithmorum Chilias Prima* of HENRY BRIGGS. This is the earliest publication of logarithms to the base 10; it exhibits the logarithms to 14D of numbers 1(1)1000. Table F was given in part VIII.

Dr. THOMPSON's splendid Introduction to *Logarithmetica Britannica* in part II includes a discussion of: (a) Interpolation with examples (p. xv–xxviii); (b) Method of Construction (p. xxviii–lxii)—here are 12 special tables and a plate illustrating the unique four-bank integrating and differencing machine which Thompson built for carrying through the calculations

of this work. He personally set up the type for all table entries by means of a monotype keyboard. This system of type setting involves the use of two entirely separate machines, a keyboard and a typecaster. Thompson used the keyboard to punch holes in a continuous ribbon of specially prepared paper, which then went to the typecaster.

There is a section (p. lvi–lxii) on construction of a table of Anti-logarithms. Shortly after the publication of the table of logarithms had begun, Thompson was urged to undertake a companion volume of anti-logarithms on the same scale. Table 8 (p. lviii) is a single page of such a table log $N = 0.00(.01)1.00$, corresponding N's being given to 28D.

"Printing and proof-reading" and "Acknowledgments" on p. lxii–lxv are followed by "References" (p. lxv–lxvi) listing works mentioned in the Introduction, and a few other titles.

In the manuscript for the main table the calculations were made to about 24D, the last digit having a possible error of three or four units. In starting computations it was desirable to have a considerable number of values of log N to a large number of decimal places. For one thing Dr. THOMPSON checked by 63D computation the accuracy of the 61D table of log N, $N = 1(1)100$ and primes to 1097; also 999990(1)1000110, in ABRAHAM SHARP'S *Geometry Improv'd* (1717). The figures in this table are arranged in five-figure groups except the twelfth and last group, which consists of six figures. The following errors in mantissas were found (the error for $N = 751$ was new):

N	Group	*For*	*Read*
103	9	33496	23496
227	12	494656	495656
751	12	287788	287771
839	12	539741	538741
1009	12	382385	382285

No other errors were found except for unit errors in the 61st decimal place. For $N = 127, 149, 293$, the final digits should each be increased by unity.

Thompson had only the CALLET (1795) reprint by Sharp, and found that Callet had an error for $N = 1097$; but here Sharp was correct. In the PETERS & STEIN reprint (1922) all of Callet's errors persist.

Other extensive log N tables are: (a) A. GRIMPEN'S 84D table (1922) of the prime numbers up to 113; and (b) H. M. PARKHURST'S 100D table (1876) for 96 values of $N \leqslant 109$. See *MTAC*, v. 1, p. 20, 58–59, 121–122.

Appendix (i) of part II contains the first English translation of the earliest published biography of BRIGGS, in THOMAS SMITH'S *Vitae quorundam eruditissimorum et illustrium virorum*, 1707. There is no known portrait of Briggs (see *MTAC*, v. 2, p. 287; v. 3, p. 67).

Appendix (ii) lists the errors in *Arithmetica Logarithmica* (1624), referred to above.

Then follow: Table F (lxxxv–xciv); Table G, antilogarithms of logarithms [0000000(1)0000450; 21D], p. xcv–xcvii; Table H, Short Tables and Constants, p. xcviii; and finally (p. xcix–cv) log N, N = [1(1)1000; 21D].

The nine parts of this work have been published in *Tracts for Computers* prepared in the Department of Statistics, University of London, in the following numbers: XIX, XXII, XXI, XVI, XVII, XVIII, XX, XIV, XI.

Dr. THOMPSON tells us that during the past 30 years no error has been reported for any of nearly 3000000 figures in the eight parts published up to 1937. We tender the author our heartiest felicitations on his truly monumental personal contributions achieved in producing this great work.

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1098[F].—H. J. A. DUPARC, C. G. LEKKERKERKER & W. PEREMANS, *Reduced sequences of integers and pseudo-random numbers*, Math. Centrum, Amsterdam, *Report* ZW1952-002, 15 mimeographed leaves.

This study of the length of the period of a geometric progression or a Fibonacci sequence when the terms are reduced modulo m, contains the following small tables.

P. 3 contains a table of the factors of $10^n - 1$, Euler's totient function of these factors and their least common multiple for $n = 1(1)10$. The corresponding information for $10^n + 1$ is given for $n = 1(1)7$.

P. 4 gives the same information for $2^n \pm 1$ for $n = 1(1)16, 29, 30$.

P. 15 gives the rank of apparition of p in the Fibonacci sequence 0, 1, 1, 2, 3, 5, $\cdots$, for each prime less than 433. This table was taken from the table of JARDEN.[1]

D. H. L.

[1] D. JARDEN, "Table of the ranks of apparition in Fibonacci's sequence," *Riveon Lematematika*, v. 1, no. 3, 1946, p. 54 [*MTAC*, v. 2, p. 343].

1099[F.]—KARL GOLDBERG, "A table of Wilson quotients and the third Wilson prime," London Math. Soc., *Jn.*, v. 28, 1953, p. 252–256.

The Wilson quotients W_p are defined as the non-negative residues modulo p of $[(p + 1)! + 1]/p$, where p is prime and the Wilson primes are solutions of the equation $W_p = 0$. The present table gives the values of W_p for all primes less than 10000, and shows 563 to be the third Wilson prime.

N. G. W. H. BEEGER's table[1] of Wilson Quotients extended to all primes less than 300 and the previously known Wilson primes were 5 and 13.

GOLDBERG states that six months after the completion of this table DONALD WALL [UMT **150**, *MTAC*, v. 6, p. 238] computed a table of Wilson quotients for all primes less than 5000, and that his table checked with this one in every case.

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[1] N. G. W. H. BEEGER, "On the congruence $(p - 1)! \equiv -1 \pmod{p^2}$," *Messenger Math.*, v. 49, 1920, p. 177–178.

1100[F].—F. GRUENBERGER, *Table of Prime Numbers from* 2 *to* 406253. Numerical Analysis Laboratory, Univ. of Wisconsin, Madison, 1953, 1 "microcard," 7.6 × 12.4 cm. Price, 25 cents.

This little card contains 34320 primes, each prime fully spelled out, as photographed from 65 sheets, calculated and printed with an IBM Card Programmed Calculator.

The condensation achieved is remarkable. Twenty cards of this size would accommodate the 665000 primes in LEHMER's list[1] covering the first 10 million numbers.

The card requires a hand lens of power $\geqslant 5$. The time required to enter the table is nearly the same as for a standard table, the time spent in locating the appropriate region of the card being comparable with that spent in turning pages.

The list was compared at 520 places (every 66th prime) with that of Lehmer.

D. H. L.

[1] D. N. LEHMER, *List of Prime Numbers from* 1 *to* 10006721, Washington, 1914.

1101[F].—GIUSEPPE PALAMÀ & L. POLETTI, "Tavola dei numeri primi dell'intervallo 12 012 000 – 12 072 060," Unione Matematica Italiana, *Bollettino*, s. 3, v. 8, 1953, p. 52–58.

With references to the D. N. LEHMER list of primes up to 10 006 721 (1914); and to the list of N. G. W. H. BEEGER, L. POLETTI, A. GLODEN & R. J. PORTER, from 10 006 741–10 999 997 (1951, *MTAC*, v. 6, p. 81–82); the authors add 3684 primes before 12 072 060.

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1102[F].—G. RICCI, "La differenza di numeri primi consecutivi," Univ., Politec., Torino, *Rend. Sem. Mat.*, v. 11, 1952, p. 149–200.

This topical history contains the following tables and graphs.

Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$ be the primes in increasing order and let the difference between consecutive primes be denoted by

$$d(n) = p_{n+1} - p_n.$$

Let $A_{2h}(x)$ denote the number of primes $p \leqslant x$ for which $p + 2h$ is also a prime. Let $B_{2h}(x)$ denote the number of those primes $p_n \leqslant x$ for which $d(n) \leqslant 2h$.

The main table (p. 152–155) gives the values of

$$p_n,\ d(n),\ A_{2h}(p_n),\ (h = 1(1)7);\quad B_{2h}(p_n),\ (h = 1(1)5)$$

for $n = 1(1)170$. The function $d(n)$ is graphed on p. 157 and compared with $\ln n$. The functions

$$A_{2s}(x),\ (h = 1(1)7)$$

and the function

$$A(x) = d(n) \quad (p_n \leqslant x < p_{n+1})$$

are graphed on p. 158–9 for $0 < x < 1020$.

WESTERN's[1] table of those primes $p_n < 10^7$ whose difference $d(n)$ exceeds that of all smaller primes is reprinted.

D. H. L.

[1] A. E. WESTERN, "Note on the magnitude of the difference between successive primes," London Math. Soc., *Jn.*, v. 9, 1934, p. 276–278.

1103[I,L].—NBS *Tables of Chebyshev Polynomials $S_n(x)$ and $C_n(x)$*. Applied Math. Ser. No. 9. U.S. Government Printing Office, Washington, D. C., 1952. xxix + 161 p., 20.5 × 27 cm. $1.75.

"The Chebyshev polynomials are of use in many mathematical investigations. Although direct numerical tabulation is fairly easy to avoid—for example, by double or multiple use of ordinary trigonometrical tables—the present tables are welcome because they will remove the necessity for these roundabout methods, which are often irritating" (from the Foreword by J. C. P. MILLER).

The polynomials tabulated in this volume are

$$C_n(x) = 2\cos n\,\theta$$

$$S_n(x) = \frac{\sin (n+1)\,\theta}{\sin\theta}$$

where

$$x = 2\cos\theta.$$

Other notations are

$$T_n(x) = \frac{1}{2}C_n(2x), \; T_n{}^*(x) = \frac{1}{2}C_n(4x-2)$$

$$U_n(x) = S_n(2x), \qquad U_n{}^*(x) = S_n(4x-2).$$

Explicit expressions are given for $C_n(x)$, $S_n(x)$, $T_n(x)$, $U_n(x)$ for $n = 0\,(1)\,12$; and for $T_n{}^*(x)$ for $n = 0\,(1)\,20$. The expansions of x^k in the $T_n(x)$, and in the $T_n{}^*(x)$, polynomials are also given for $k = 0\,(1)\,12$.

The principal tables give 12D values of $S_n(x)$ and $C_n(x)$ for $n = 2\,(1)\,12$, $x = 0\,(.001)\,2$.

The Introduction, by CORNELIUS LÁNCZOS, describes the basic properties of Chebyshev polynomials, their applications to expansions, curve fitting, solution of linear differential equations with rational coefficients; it gives an account of the computation of the tables and instructions for their use, and also a list of references.

The computations were carried out under the technical direction of A. N. LOWAN.

A preliminary MS of these tables was described by LOWAN in *MTAC*, v. 1, p. 125 (UMT 11). Other tables of Chebyshev polynomials are referred to in *MTAC*, v. 1, p. 385 (RMT **185**), v. 2, p. 256 (RMT **371**), v. 2, p. 262–263 (RMT **381**), v. 2, p. 266 (RMT **383**), v. 3, p. 97 (RMT **495**), v. 3, p. 119 (MTE **126**), v. 3, p. 120–121 (UMT **68**).

"The tabulation of these polynomials—easy to calculate, and easy to sidetrack at the cost of some inconvenience—is long overdue. The Computation Laboratory staff is to be congratulated on the removal, at last, of this source of inconvenience, and in doing so, on the addition of yet one more table to its already magnificent series" (from the Foreword).

A. E.

1104[I].—H. E. SALZER, "Formulas for numerical differentiation in the complex plane," *Jn. Math. Phys.*, v. 31, 1952, p. 155–169.

In an earlier paper[1] the author gave coefficients for numerical integration in the complex plane of polynomials of degree no higher than eight, based on

configurations of the grid points chosen so as to be convenient for initiating a computation and "as close together as possible." Thus let the Lagrange polynomial of degree $n - 1$ be defined by

$$(1) \qquad f(z) = \sum_{k=1}^{n} P_k(z) f(z_k)$$

where $f(z_k)$ is the given value of $f(z)$ at $z_k = x_k + iy_k$. The points are chosen over a *square* grid, namely

$$z_j = z_0 + jh$$

where j is a complex integer. As an example, for a polynomial of degree four the selected grid-points are

$$z_0, z_0 + h, z_0 + 2h, z_0 + ih, z_0 + (i + 1)h.$$

The s-th derivative of $f(z)$ at any grid point z_j can be expressed by

$$(2) \qquad f^{(s)}(z_j) = \sum_k P_k^{(s)}(z_j) f(z_k).$$

Letting $z = z_0 + Ph$, it is possible to write

$$(3) \qquad h^s f^{(s)}(z_j) = \sum_k M_k^{(s)}(j) f_k / M(s).$$

where $f_k = f(z_k)$, and $M_k^{(s)}(j)$ and $M(s)$ are complex integers, independent of f and h. The author tabulates the exact coefficients of f_k for all grid points, in the derivatives of all orders, for polynomials of degree 2, 3, $\cdots$, 9. Methods of computing and checking are described in the paper. The work will add to the author's reputation for supplying accurate and well planned tables of coefficients.

GERTRUDE BLANCH

NBSINA

[1] H. E. SALZER, "Formulas for numerical integration of first and second order differential equations in the complex plane, *Jn. Math. Phys.*, v. 29, 1950, p. 207–216.

1105[K].—E. P. KING, "The operating characteristic of the control chart for sample means," *Annals Math. Stat.*, v. 23, 1952, p. 384–395.

This paper extends the theory of the SHEWHART control chart by deriving expressions for the chances that the $\bar{X}$ chart or chart for sample means will show control for both of the cases of known and unknown standard deviation. The null-hypothesis is that samples are drawn from the usual normal process, $N(\mu, \sigma^2)$, where μ and σ are fixed but unknown. The alternative hypothesis considered is that shifts in the process mean from time to time can be represented by $N(\mu, \theta^2\sigma^2)$, i.e., the process mean itself is a random variable with this normal distribution.

Suppose we consider m samples of n each which are selected at random from rational categories of a process and we plot the $\bar{X}$ and R (sample range) charts. Let $\beta_0(k, \theta, m, n)$ denote the chance that all m sample means will fall within $\pm k\sigma n^{-\frac{1}{2}}$ of the grand mean, where the standard deviation σ is known, and let $\beta(k, \theta, m, n)$ represent the chance that the $\bar{X}$ chart will show control when σ, in effect, is estimated from the m sample ranges on the chart for ranges, i.e., the process itself to date. Of course, it is usual practice to take $k = 3$ and the computed tables of the paper are based on this value. The Tables I–VI (p. 390–392) give values to 2D for the operating character-

istics or the chances, β, that the $\bar{X}$ chart will show control for several practical cases: $\beta_0(3, \theta, m, n)$ for $\theta = 0(.5)3$; $n = 2, 5, 10$, $m = 2, 3, 4$; $\beta(3, \theta, m, n)$ for $\theta = 0(.5)3$; $n = 2, 5, 10, m = 2$, and $n = 5, 10, m = 3, 4$.

For $m \geq 4$ bounds for $\beta_0(3, \theta, m, n)$ and $\beta(3, \theta, m, n)$ are given in Tables VII and VIII (p. 393, 394) to 2D for the cases: $\theta = 0(.5)3$; $n = 5, 10$; $m = 5, 10$; $\theta = 0, .75, 1, 1.25, 1.5$; $n = 5, 10$; $m = 15, 20$; $\theta = 0, .25, .75, 1$; $n = 5, 10$; $m = 25$.

It is of interest to note that the problem treated in this paper is somewhat related to that of testing for "outlying" observations, since the chance that the $\bar{X}$ chart will show control is also the chance that the largest and smallest sample means will both lie between control limits.

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1106[K].—W. H. KRUSKAL & W. A. WALLIS, "Use of ranks in one-criterion variance analysis," Amer. Stat., Assn., *Jn.*, v. 47, 1952, p. 583–621.

Consider three samples, of sizes n_1, n_2, and n_3 respectively, arranged together in order of size. Assign scores to the $N = n_1 + n_2 + n_3$ individuals according to their ranks in the combined sample. That is, assign the score 1 to the smallest of the N, the score 2 to the next smallest, etc. Let $H = \frac{N-1}{N} \sum_{i=1}^{3} \frac{n_i[R_i - \frac{1}{2}(N+1)]^2}{(N^2-1)/12}$, where R_i is the mean rank score of the ith sample.

Under the assumption that the three samples are random selections from a continuous population Table 6.1 (p. 614–617) of this paper gives, for n_1, n_2, $n_3 \leq 5$, exact probabilities (and three approximations to the probabilities) all to 3D, that H will equal or exceed certain selected values. The selected values are chosen so that the probabilities will be close to 10, 5, and 1 per cent.

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1107[K].—L. E. MOSES, "A two-sample test," *Psychometrika*, v. 17, 1952, p. 239–247.

Samples of size m and n are drawn from populations A and B, respectively, and the combined samples are arranged in increasing order. To test the hypothesis that $A = B$ against the alternative that B is more widely dispersed than A, the author proposes the statistic s_h^*, defined as one more than the difference in the ranks of the h-th largest and h-th smallest observation in the first sample. He tables (p. 244, 245) the tail of the distribution of s_h^* to 3D for $h = 3, 4, 5, 8$, $m = 4h$, and 5 suitably selected values of n in each case.

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1108[K].—P. J. RIJKOORT, "A generalization of Wilcoxon's test," Nederlandsche Akademie van Wetenschappen, *Proceedings*, v. 55, series A, 1952, p. 394–404.

Let x_{ij} be the j-th observation in the i-th sample, where $i = 1, \cdots, k$ and $j = 1, \cdots, n_i$. Let r_{ij} be the rank of x_{ij} in the combined sample of $n = \sum n_i$ observations. The author proposes testing the hypothesis that all of the samples come from distributions with the same mean value by the use of the statistic

$$S = \sum (s_i - n_i\bar{r})^2$$

where $s_i = \sum r_{ij}$ and $\bar{r} = (n + 1)/2$.

The tables (p. 400–402) give to 3 and 4D (sometimes more) the exact cumulative distribution function of S: for $k = 3$ and each $n_i = 2, 3$, or 4; $k = 4$ and each $n_i = 2$; $k = 5$ and each $n_i = 2$, for all values of S, and for $k = 3$ and each $n_i = 5$; $k = 4$ and each $n_i = 3$, for large values of S.

Methods of approximating the distribution function of S by the chi square distribution and the analysis of variance distribution are presented. Using these methods a table (p. 402) is given showing the approximate five per cent points of S for all combinations of $k = 3(1)10$ and $n_i = 2(1)10$.

The statistic S is a linear function of the statistic H proposed by KRUSKAL & WALLIS.[1] These authors give tables of the five and one per cent points for H for $k = 3$ and all possible combinations of $n_i \leq 5$.

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[1] W. H. KRUSKAL & W. A. WALLIS, "Use of ranks in one-criterion variance analysis," Amer. Stat. Assn., *Jn.*, v. 47, 1952, p. 583–621. (RMT 1106.]

1109[K].—COLIN WHITE, "The use of ranks in a test of significance for comparing two treatments," *Biometrics*, v. 8, 1952, p. 33–41.

This paper presents tables for use of the WILCOXON procedure[1] for comparing two treatments when the numbers of individuals in the two groups are not necessarily equal. Let n_1 be the number of individuals in the group for which we compute the rank total T while n_2 is the number of individuals in the other group. Without loss of generality, n_1 can always be taken less than or equal to n_2. The ranks allotted are $1, 2, \cdots, (n_1 + n_2)$. The null hypothesis tested is that both treatments had the same effect; that is, that T represents the sum of n_1 ranks drawn at random from the finite universe $1, 2, \cdots, (n_1 + n_2)$. Let the integer T_α (notation of reviewer) have the property that $\Pr(T \leq T_\alpha) \leq \alpha$ and $\Pr(T \leq T_\alpha + 1) > \alpha$. Then also $\Pr[T \geq n_1(n_1 + n_2 + 1) - T_\alpha] \leq \alpha$ and $\Pr[T \geq n_1(n_1 + n_2 + 1) - T_\alpha - 1] > \alpha$. Tables of T_α (p. 37–39) are presented for $\alpha = 5\%, 1\%, .1\%$ and all n_1, n_2 up to $n_1 + n_2 \leq 30$.

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[1] F. WILCOXON, "Individual comparisons by ranking methods," *Biometrics*, v. 1, 1945, p. 80–83.

1110[L].—Le Centre National d'Études des Télécommunications, *Tables des fonctions de Legendre associées*. Editions de la Revue d'Optique, Paris, 1952. xxi + 292 p., 21 × 30.5 cm.

Legendre functions whose degree is not an integer, arise in potential and wave problems relating to cones. Some computations of such functions have been reviewed previously (*MTAC*, v. 5, p. 152–153; v. 6, p. 98–99; see also RMT **1120**) but no systematic tabulation seems to exist, apart from the case when the degree is an integer or half of an odd integer. Thus the present tables are a pioneering effort in a field which is becoming more and more important.

The function tabulated in this volume is

$$P_n^m(\cos\theta) = (-\sin\theta)^m \frac{d^m P_n(\cos\theta)}{d(\cos\theta)^m}$$

where m is a non-negative integer, θ is real, and n is real.

The introductory material includes a Preface by J. Coulomb; an Introduction by L. Robin; an account of the method of computation, checks, accuracy by P. Le Gall; a diagram facilitating the use of the recurrence formulas; the level curves of $P_n^0(\cos\theta) = \text{const.}, \cdots, P_n^5(\cos\theta) = \text{const.}$ in the region $0 \leq n \leq 10$, $0° \leq \theta \leq 90°$ of the n, θ plane; and a detailed page index.

The tables give values of $P_n^m(\cos\theta)$, to various degrees of accuracy, for $m = 0\,(1)\,5$, $n = -.5\,(.1)\,10$, $\theta° = 0°\,(1°)\,90°$. A second volume (in preparation) will extend the tables to $\theta = 180°$.

A. E.

1111[L].—P. C. Clemmow & Cara M. Munford, "A table of $\sqrt{(\frac{1}{2}\pi)}\, e^{\frac{1}{2}i\pi\rho^2} \int_\rho^\infty e^{-\frac{1}{2}i\pi\lambda^2}\, d\lambda$ for complex values of ρ." R. Soc. of London, *Phil. Trans.*, v. 245A, 1952, p. 189–211.

Tables of the error function for complex variable were long overdue. They are needed in many wave propagation, and in some other problems. The computation of this function was discussed in *MTAC*, v. 5, p. 67–70.

The function chosen for tabulation in this memoir is

$$G(\rho) = \sqrt{(\tfrac{1}{2}\pi)}\, e^{\frac{1}{2}i\pi\rho^2} \int_\rho^\infty e^{-\frac{1}{2}i\pi\lambda^2}\, d\lambda.$$

For large ρ with $0 \leq \arg\rho \leq \frac{1}{4}\pi$ it is asymptotically represented by

$$\frac{1}{i\sqrt{(2\pi)}\rho}\left[1 - \frac{1}{i\pi\rho^2} + \frac{1.3}{(i\pi\rho^2)^2} - \cdots\right]$$

and

$$G(0) = \tfrac{1}{2}\pi^{\frac{1}{2}} e^{-\pi i/4}.$$

The table gives 4D values of the real and the imaginary parts of $G(\rho)$ for $\rho = re^{i\theta}$ where $r = 0\,(.01)\,.8$, $\theta° = 0°\,(1°)\,45°$.

A. E.

1112[L].—H. M. DAGGETT, JR., "The Shedlovsky extrapolation function," Amer. Chem. Soc., *Jn.*, v. 73, 1951, p. 4977.

4D tables of

$$\{\tfrac{1}{2}z + [1 + (\tfrac{1}{2}z)^2]^{\frac{1}{2}}\}^2$$

for $z = 0(.001).209$.

A. E.

1113[L].—A. A. DORODNIT͡SYN, "Asimptoticheskie zakony raspredeleniia sobstvennyk͡h znachenii dlia nekotoryk͡h osobyk͡h vidov different͡sialnyk͡h uravnenii vtorogo poriadka." [Asymptotic laws of distribution of the characteristic values for certain special forms of differential equations of the second order.] *Uspekhi Matem. Nauk* (N.S.) v. 7, no. 6, 1952, p. 3–96.

Two short tables (p. 95) give numerical values to 4–6 S of

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}$$

Table I. $s = 2(2)8$, $a = 3(1/12)4$.

Table II. $s = 2(2)8$, $a = 3(.05)4$.

A. E.

1114[L].—H. GÖRTLER, "Zur laminaren Grenzschicht am schiebenden Zylinder. Teil I." *Arch. Math.*, v. 3, 1952, p. 216–231.

The functions $F_0, \cdots, F_{222}$ are solutions of the differential equations

$$F_0'' + f_1F_0' = 0$$
$$F_2'' + f_1F_2' - 2f_1'F_2 = -12f_3F_0'$$
$$F_4'' + f_1F_4' - 4f_1'F_4 = -30g_5F_0'$$
$$F_{22}'' + f_1F_{22}' - 4f_1'F_{22} = -30h_5F_0' - 12f_3F_2' + 8f_3'F_2$$
$$F_6'' + f_1F_6' - 6f_1'F_6 = -56g_7F_0'$$
$$F_{24}'' + f_1F_{24}' - 6f_1'F_{24} = -56h_7F_0' - 30g_5F_2' + 12g_5'F_2 - 12f_3F_4' + 16f_3'F_4$$
$$F_{222}'' + f_1F_{222}' - 6f_1'F_{222} = -56k_7F_0' - 30h_5F_2' + 12h_5'F_2 - 12f_3F_{22}' + 16f_3'F_{22}.$$

All F's vanish at 0, $F_0 = 1$, and all the other F's vanish at ∞. The f_i, g_i, h_i, k_i are the functions appearing in the integration of the boundary layer equations according to BLASIUS[1] and HOWARTH.[2]

The present paper gives 3D or 4D tables and diagrams of $F_0(\eta)$, $F_0'(\eta)$, $\cdots$, $F_{222}(\eta)$, $F_{222}'(\eta)$ for $\eta = 0(.1)5.4$ (except for $F_{24}, \cdots, F_{222}'$ when $\eta \leq 5.2$). The values have been computed by numerical integration of the differential equations. The necessary values of $f_i, \cdots, k_i$ were taken from the tables by ULRICH,[3] and subtabulated where necessary. The values of F_0 and F_0' were compared with those given by COOKE[4] and SCHLICHTING;[5] discrepancies in the last decimal were noted.

A. E.

[1] H. BLASIUS, "Grenzschichten in Flüssigkeiten mit kleiner Reibung," *Zschr. f. Math. u. Phys.*, v. 56, 1907, p. 1–37.

[2] L. HOWARTH, *Steady flow in the Boundary Layer near the Surface of a Stream.* ARC *Report* No. 1632, 1934.

[3] A. ULRICH, "Die ebene laminare Reibungschicht an einem Zylinder," *Arch. d. Math.*, v. 2, 1949, p. 37–41; *MTAC*, v. 4, 1950, p. 96–97.
[4] J. C. COOKE, "The boundary layer of a class of infinite yawed cylinders," Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 645–648.
[5] H. SCHLICHTING, *Grenzschicht-Theorie*. G. Braun, Karlsruhe, 1951.

1115[L].—J. L. LUBKIN & Y. L. LUKE, "Frequencies of longitudinal vibration for a slender rod of variable section," *Jn. Appl. Mech.*, v. 20, 1953, p. 173–177.

The frequency equation is

$$[J_1(\alpha)Y_1(\beta) - Y_1(\alpha)J_1(\beta)]\sin\gamma + (\epsilon/|\epsilon|)[J_1(\alpha)Y_0(\beta) - Y_1(\alpha)J_0(\beta)]\cos\gamma = 0$$

where $\alpha = (1+\epsilon)\beta$, $\beta = \nu_n L_2/|\epsilon|$, $\gamma = \nu_n L_1$ and $\nu_1, \nu_2\cdots$ are proportional to the frequencies. The dimensionless parameters ϵ and $\lambda = L_1/(L_1+L_2)$ are used, and table 1 gives 5 D values of $\nu_n(L_1+L_2)$ for $n = 1\,(1)\,5$, $\epsilon = -1\,(1/3)\,1$, $\lambda = 0\,(.125)\,1$. "Seven or eight decimals are carried in the computations, sufficient to insure that only occasional entries, if any, are in error by as much as one unit in the fifth place."

Table 2 gives some auxiliary quantities, useful in interpolating in the ϵ-direction.

A. E.

1116[L].—D. MANTERFIELD, J. D. CRESSWELL, & H. HERNE, "The quick-immersion thermo-couple for liquid steel," Iron and Steel Institute, *Jn.*, v, 172, 1952, p. 387–402.

Table I, p. 396, gives 4D values of the first six roots x of the equation

$$J_0(kx)Y_1(x) - Y_0(kx)J_1(x) = 0$$

for $k = (1.1)^{\frac{1}{2}}\ 1.06(.02)\ 1.1(.05)\ 1.3(.1)\ 1.5,\ 2(1),\ 5.$

Table II gives the corresponding 4D values of

$$\frac{2J_1(x)J_0(kx)}{x\{[J_0(kx)]^2 - [J_1(x)]^2\}}$$

for $k = (1.1)^{\frac{1}{2}},\ 1.15(.05)\ 1.3(.1)\ 1.5,\ 2(1)\ 5.$

The tables were computed by H. CARSTEN and N. MCKERROW; they are referred to in FMR *Index*, section 17.812, p. 268.

A. E.

1117[L].—NBS, Applied Mathematics Series No. 13, *Tables for the Analysis of Beta Spectra.* U. S. Government Printing Office, Washington, D. C., 1952. iii + 61 p., 20 × 26.5 cm. \$0.35.

These tables were designed to assist in the theoretical analysis of beta-ray spectra.

The principal tables give values of the "Fermi function"

$$f(Z,\eta) = \eta^{2+2S}\, e^{\pm\pi\delta}|\Gamma(1+S+i\delta)|^2$$

where Z is the atomic number, η the momentum of the electron,

$$\epsilon = (1+\eta^2)^{\frac{1}{2}}, \beta = \eta/\epsilon, \gamma = Z/137, S = (1-\gamma^2)^{\frac{1}{2}} - 1, \delta = \gamma/\beta.$$

Other abbreviations used in this pamphlet are

$$T = 510.91(\epsilon - 1), B\rho = 1704.3\eta.$$

A brief introduction gives definitions and indicates the method of computations, and is followed by sections on beta spectra and their analysis, and a bibliography.

There is a set of 6 auxiliary tables.

Table 1 (p. 15–16). Values of η, ϵ, β, T[4S or 5S) for $B\rho = 0(100)20000$ Gauss cm.

Table 2 (p. 17). 5S values of ϵ, η, β, $B\rho$ for $T = 0$ (10) 60 (20) 200 (50) 1000(100) 5000 (200) 10000 kev.

Table 3 (p. 18). Values of the "nonrelativistic" electrostatic effect factor.

Table 4 (p. 18). Illustrating the importance of the relativistic effect for $Z = 0, 10$ (20) 70, 100 and $\eta = 0, .2, .6, 1, 2, 4, 7$.

Table 5 (p. 18). Values of the ratio

$$\left|\frac{\Gamma(1 + S + i\delta)}{\Gamma(1 + i\delta)}\right| (\delta^2 + \tfrac{1}{4})^{-S}$$

for the same values of Z, η as in Table 4 except that $\eta = 0$ is omitted.

Table 6 (p. 18, 19). Values of $\dfrac{1}{5.109}\dfrac{\partial}{\partial\sqrt{1 + \eta^2}}\dfrac{\sqrt{1 + \eta^2}}{\eta}f(Z, \eta)$ to 2 or 3S and of $\dfrac{2\pi\gamma}{5.109\eta^3}$ to 4D (two separate tables) for $Z = 0, 10, 20$ (20) 100, $\eta =$ 0 (.2) 1 (1) 6.

Note that the "Fermi-function" $f(Z, \eta)$ is not identical with the "Fermi-Dirac function" $F_n(\eta)$.

The principal table (p. 21–61) gives values of $f(Z, \eta)$ for $Z = 1$ (1) 100, $\eta = 0$ (.05) 1 (.1) 7. Each Z has a column to itself going over two adjacent pages. The column heading gives values of Z, γ, S, $\varphi(Z)$, and each row (for a given η, T, ϵ, $B\rho$ all of which are stated) gives the values of $f(Z, \eta)$ indicated as β^- and β^+ for the upper and lower signs in the exponential respectively. At the foot of each column constants A,B,C are listed, and computation from the approximate formula

$$f(Z, \eta) \approx \eta^{2+2S}\left(A + \frac{B}{\eta} + \frac{C}{\eta^2}\right)$$

is recommended for $\eta = 0$.

On p. 10, and again on p. 21, $\varphi(Z)$ is defined as

$$(4\pi mcR/h)^{2S}[\Gamma(3)/\Gamma(3 + 2S)]^2(1 + S/2),$$

and this definition was used in the auxiliary tables. In a mimeographed correction sheet (which should accompany each copy) it is pointed out that the values listed as $\varphi(Z)$ on p. 22–61 of the principal tables are actually values of

$$\left(\frac{2 + 2S}{3 + 2S}\right)^2 \varphi(Z).$$

This sheet corrects the value for $Z = 100$, and lists 5S values of $\varphi(Z)$ for $Z = 1$ (1) 100.

A. E.

1118[L].—NBS, Applied Mathematics Series No. 25, *Tables of Bessel functions* $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leq x \leq 1$. U. S. Government Printing Office, Washington, D. C., 1952. ix + 60 p., 20 × 26.5 cm. $0.40.

This is a reprint of *Applied Mathematics Series*, No. 1 and was reviewed in *MTAC*, v. 3, p. 187–188.

A. E.

1119[L].—NBS, Applied Mathematics Series, No. 28, *Tables of Bessel-Clifford functions of orders zero and one.* U. S. Government Printing Office, Washington, D. C., 1953. ix + 72 p., 20 × 26.5 cm. $0.45.

"Although the Bessel-Clifford functions are obtainable from existing tables of Bessel functions, it was felt that they warranted tabulation because it is generally necessary to enter the existing tables with an irrational argument. Furthermore, the Bessel-Clifford functions arise as solutions of a class of differential equations occurring in various branches of applied physics, and they are therefore of importance in themselves." "The present volume carries the tabulation of the functions of orders zero and one up to a point where the asymptotic expansions can be used conveniently." (From the introduction.)

Table I. $J_0(2\sqrt{x})$ for $x = 0$ (.02) 1.5 (.05) 3 (.1) 13 (.2) 45 (.5) 115 (1) 410, 8D. $J_1(2\sqrt{x})/\sqrt{x}$ for $x = 0$ (.02) 1.5 (.05) 3 (.1) 13 (.2) 45 (.5) 115 (1) 125, 8D, $x = 125$ (1) 410, 9D.

Table II. $Y_n(2\sqrt{x})(\sqrt{x})^{-n}$, same values of n and x as in Table I.

Table III. $I_0(2\sqrt{x})$ for $x = 0$ (.02) 1, 8D, $x = 1$ (.02) 1.5 (.05) 6.2, 7D. $I_1(2\sqrt{x})/\sqrt{x}$ for $x = 0$ (.02) 1.5 (.05) 6.2, 7D.

Table IV. $e^{-2\sqrt{x}}\, I_0(2\sqrt{x})$ for $x = 6.2$ (.1) 13 (.2) 36 (.5) 115, 8D, $x = 115$ (1) 160 (5) 410, 9D. $e^{-2\sqrt{x}}\, I_1(2\sqrt{x})/\sqrt{x}$ for $x = 6.2$ (.1) 13 (.2) 36 (.5) 65, 8D, $x = 65$ (.5) 115 (1) 160 (5) 410, 9D.

Table V. $K_0(2\sqrt{x})$ for $x = 0$ (.02) 1.5 (.05) 2.5, 8D, $x = 2.5$ (.05) 6.,2 9D. $K_1(2\sqrt{x})/\sqrt{x}$ for $x = 0$ (.02) .04, 6D, $x = .06$ (.02) .28, 7D, $x = .30$ (.02) 1.5 (.05) 2.5, 8D, $x = 2.5$ (.05) 6.2, 9D.

Table VI. $e^{2\sqrt{x}}\, K_0(2\sqrt{x})$ for $x = 6.2$ (.1) 13 (.2) 36 (.5) 115 (1) 160 (5) 410, 8D. $e^{2\sqrt{x}}\, K_1(2\sqrt{x})/\sqrt{x}$ for $x = 6.2$ (.1) 13 (.2) 36 (.5) 40, 8D, $x = 40$ (.5) 115 (1) 160 (5) 410, 9D.

Except in the neighborhood of singularities, second central differences, sometimes modified, are given in all six tables.

Auxiliary tables. Interpolation coefficients for Everett's formula.

The Introduction (by M. ABRAMOWITZ) gives the mathematical properties of Bessel-Clifford functions, describes (mathematical) applications, interpolation, the method of computation, and accuracy, and lists references.

A preliminary tabulation of these functions was reported in *MTAC*, v. 3, p. 107.

A. E.

1120[L].—K. M. Siegel, J. W. Crispin, R. E. Kleinman, & H. E. Hunter, "The zeros of $P_{n_i}^{(1)}(x_0)$ of non-integral degree." *Jn. Math. Phys.*, v. 31, 1952, p. 170–179.

Let $P_n^{(1)}(x)$ be the associated Legendre function of order unity and non-integral degree. This paper is concerned with determination of n_i and $\int_{x0}^{1} [P_{n_i}^{(1)}(x)]^2\,dx$ such that $P_{n_i}^{(1)}(x) = 0$. The evaluation of n_i has previously been considered (*MTAC*, v. 5, p. 152–153; v. 6, p. 98–99). The idea of the paper is to expand $P_{n_i}^{(1)}(x)$, $n_i = n + Z_n$, in a Taylor series and calculate Z_n by inversion. The formula for Z_n depends on Legendre and associated Legendre functions and a certain sum. The latter is finite if n is an integer or an odd multiple of 1/2, and for these choices of n computation is facilitated as many tables exist for the former elements. The series expansion is also used to evaluate the above intergral.

The theory is illustrated for $x_0 = \cos 165°$. In Table 3, values of $P_n(x_0)$ and $P_{n_i}^{(1)}(x_0)/\sqrt{1 - x_0^2}$ are tabulated to 10*S* for $n = 1\,(1)\,20$. Values are also given for $n = -0.5(1.0)20.5$; the number of significant figures varies from 5 to 11. These were calculated using series expansions and the authors state that existing tables were used as checks. The entries for n an odd multiple of 1/2 are mostly new. For n an integer, many of the entries are already available (cf. FMR *Index*, for example).

In Table 4, the first 19 values of n_i and $\int_{x0}^{1} [P_{n_i}^{(1)}(x)]^2\,dx$ are given. Three terms of the Taylor series are used. In each case computations are presented for both n an integer and an odd multiple of 1/2. No estimates of the error are given, but a procedure is defined to indicate preferred value according to the choice of n. The authors are interested in a speedy means to obtain entries within engineering accuracy, and the procedure outlined seems to fulfill this need.

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1121[L].—Friedrich Tölke, *Praktische Funktionenlehre. Erster Band. Elementare und elementare transzendente Funktionen.* Springer-Verlag. Berlin, Göttingen, Heidelberg. 1950. xii + 440 p., 20 × 27.5 cm.

The first edition of this volume appeared in 1943 and was not accessible to the reviewer. The present second edition is said to be considerably enlarged. Further volumes, on theta- and elliptic functions, hypergeometric, Bessel, and Legendre functions are in preparation. The whole work will present the more important functions in a manner suitable for their application in the engineering sciences.

The present volume contains a number of numerical tables. They are designed for use by engineers rather than professional computers. There are instructions for using these tables, but no sources or indications of the method of their computation, or of their accuracy.

Table 1 (p. 168–207), $2\pi x$, $\ln 2\pi x$, $e^{2\pi x}$, $e^{-2\pi x}$, $\sinh 2\pi x$, $\cosh 2\pi x$, $\tanh 2\pi x$, $\coth 2\pi x$, amp $2\pi x$, $\sin 2\pi x$, $\cos 2\pi x$, $\tan 2\pi x$, $\cot 2\pi x$, $\sin^* 2\pi x =$

$\sin(2\pi x - \frac{1}{4}\pi)$, $\cos^* 2\pi x$, $\tan^* 2\pi x$, $\cot^* 2\pi x$, to 4 or 5D, mostly with first differences, for $x = 0$ (.001) 1.

Table 2 (p. 208–247). $\frac{1}{2}\pi x$, $e^{\frac{1}{2}\pi x}$, $e^{-\frac{1}{2}\pi x}$, $\sin \frac{1}{2}\pi x$, $\cos \frac{1}{2}\pi x$ to 4 or 5S, for $x = 0$ (.01) 40.

Table 3 (p. 248–257). $\mathrm{Ei}(x)$, $\mathrm{Ei}(-x)$, $\mathrm{Shi}(x)$, $\mathrm{Chi}(x)$, $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$ to 3–4S, with differences, for $x = 0$ (.01) 5. Here

$$\mathrm{Ei}(\pm x) = \ln|x| + \sum_{n=1}^{\infty} \frac{(\pm x)^n}{n \cdot n!} = \mathrm{Chi}(x) \pm \mathrm{Shi}(x)$$

$$\mathrm{Si}(x) = -i\mathrm{Shi}(ix), \quad \mathrm{Ci}(x) = \mathrm{Chi}(ix).$$

Auxiliary tables on a folding inset to p. 258. $\frac{1}{2}\pi x$, $e^{\frac{1}{2}\pi x}$, $e^{-\frac{1}{2}\pi x}$, $\sin \frac{1}{2}\pi x$, $\cos \frac{1}{2}\pi x$, 3–5S, $x = .001$ (.001) .01. $2\pi x$, $e^{2\pi x}$, $e^{-2\pi x}$, $x = 0$ (1) 10. e^x, e^{-x}, $x = 1$ (1) 50.
$\frac{m\pi x}{4}, \frac{mx}{4\pi}, \frac{k\pi x}{3}, \frac{\pi x}{3k}$, 5D, $m = 1$ (1) 8, $k = 1$ (1) 4.
$m!$, $1/m!$, $m = 2$ (1) 10. $m!/(m-n)!$, $n = 1$ (1) m, $m = 1$ (1) 10. $\binom{n}{m}$, $m = 1$ (1) n, $n = 1$ (1) 15. Some useful numerical data.

Functions which occur in diffusion problems.

$$F_0(x, y) = \sum_{k=-\infty}^{\infty} e^{-k^2\pi x} \cos 2k\pi y,$$

5D, $x = 0(.01).05(.05).25(.25)1.5(.5)2.5, 4$, $y = 0(.01).5$ (p. 268–9).

$$F_1(x, y) = \sum_{k=1}^{\infty} (k\pi)^{-1} e^{-k^2\pi x} \sin 2k\pi y,$$

4D, $x = 0(.01).05(.05).5(.1)1(.2)2, 2.4, 2.8$, $y = 0(.01).5$ (p. 271–2).

$$F_2(x, y) = \frac{x}{4\pi} + \sum_{k=1}^{\infty} (-1)^k (2\pi^2 k^2)^{-1} [1 - e^{-k^2\pi x}] \cos 2k\pi y,$$

4D, $x = 0(.01).05(.05).5(.1)1(.2)2$, $y = 0(.01).5$ (p. 275–6).

$$F_3(x, y) = \sum_{k=1}^{\infty} (\tfrac{1}{2}\pi^3 k^3)^{-1} [1 - e^{-k^2\pi x}] \sin 2k\pi y,$$

4D, $x = 0(.0025).0125(.0125).125(.025).25(.05).5(.1)1(.25)2.5$,
$y = 0(.01).5$ (p. 282–4).

Legendre polynomials $P_n(x)$ $n = 0$ (1) 10, 4D, their derivatives $P_n'(x)$, $n = 3$ (1) 6, 4D, and their integrals

$$P_{n,k}(x) = \int_0^x \cdots \int_0^x P_n(x)(dx)^k, \quad n = 0(1)6, \quad k = 1, 2, 5\text{D},$$

for

$$x = 0(.001)1 \quad \text{(p. 372–440)}.$$

The volume also contains tables of integral formulas for indefinite integrals (p. 69–156).

A. E.

MATHEMATICAL TABLES—ERRATA

In this issue reference has been to errata in RMT 1097.

227.—S. M. Drach, "Cube roots of primes to 31 places," *Mess. Math.*, v. 7, 1877, p. 86–88.

N				*for*				*read*
2				399				228
3				108				110
5				830				860
7				756				760
13				921				915
17			08741	445			08739	726
19				829				830
29		26380	36360	183		26379	82105	597
37			05135	051			05185	119
41	38410	86376	34932	233	38409	74238	64260	790
47			93399	702			95958	893
59				833				789
61				163				148
67				224				226
73		63766	71392	658		63763	48908	768
83			76679	005			76675	948
89			80965	127			81074	269
101				024				023
103				767				765
127				742				743
4				087				308
25				984				923
49				419				408
121				282				283

The above list of errata is complete.

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228.—(a) F. Emde, *Tafeln Elementarer Funktionen*, Leipzig and Berlin, 1940, p. 123.

(b) M. Boll, *Tables Numériques Universelles*, Paris, 1947, p. 472.

Both these works contain a table of Langevin's function

$$L(x) = \coth x - x^{-1}.$$

Comparison of these with a recently computed unpublished table (see UMT162) reveals the following errata.

x	EMDE	BOLL	$L(x)$
0.20	.06649	.067	.066490
.30	.09941	.100	.099405+
.36	.11897	—	.118976
.48	.15759	—	.157595+
.60	.19536	.196	.195359
.72	.23210	—	.232095 –
1.36	.4058	—	.405746
1.72	.4848	—	.484858
1.86	.5121	—	.512037
1.90	.5195	.520	.519450 –
2.08	.5510	—	.550941
2.50	.6135	.614	.613567
2.60	.6265	.624	.626479
3.65	.7273	—	.727379
3.70	.7309	.731	.730953
4.00	.7507	.749	.750671
4.20	.7623	.762	.762355 –
4.50	.7780	.777	.778025 –
8.20	.8781	—	.878049

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229.—R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural and Medical Research*, 3rd ed., New York, 1948.

Table VI gives significance levels for the correlation coefficient. Errors in the table of the variance ratio previously reported (*MTAC*, v. 6, p. 35–38) have affected Table VI for $p = 0.001$ as follows:

n	*For*	*Read*
3	0.99116	0.99114
5	0.95074	0.95088
45	0.4648	0.4647
70	0.3799	0.3798

In addition, an error has been introduced at $n = 6$, where 0.92493 should be 0.92490. There are also a few "rounding errors." Two of these are worth recording because they constitute disagreements with the corresponding values of z. For $n = 13$, $r = 0.5139$ should be 0.5140, and for $n = 45$, $r = 0.2875$ should be 0.2876, both for $p = 0.05$.

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230.—K. HAYASHI, *Tafeln der Besselshen, Theta-, Kugel- und anderer Funktionen*, Berlin, 1930.

Table X of 12D values of square roots contains the following errata. The starred entries have been noted by FUKUOKA. There are 27 other errors in the 11th and 12th places.

k^2	*For*	*Read*
0,217	0,46583 15146 06	0,46583 *25879 54*
0,289	0,53758 72032 29	0,53758 720*2*2 29*
0.498	0,70554 94312 95	0,705*69 11505 75**
0,543	0,73688 53302 92	0,73688 533*70 78**
0.699	0,83606 21926 13	0,83606 219*8*6 *43*
0,787	0,88713 02040 49	0,88713 0204*6* 49
0,800	0,89442 70910 00	0,89442 7*1*910 00
0,845	0,91923 88101 03	0,91923 881*55 43*
0,917	0,95760 11173 76	0,95760 11*695 90*
2,53	1,59057 73720 59	1,5905*9* 73720 59
2,90	1,70293 86395 93	1,70293 863*6*5 93
3,17	1,78044 93812 80	1,78044 9381*4 76*
6,40	2,52982 20281 35	2,52982 2*1*281 35
7,36	2,71293 18932 50	2,71293 1*9*932 50
8,55	2,92403 83031 93	2,92403 8303*4 43*
8,71	2,94127 09126 75	2,9*5*127 09126 75
8,72	2,94296 46120 47	2,9*5*296 46120 47
8.73	2,94465 73405 39	2,9*5*465 73405 39
8.74	2,94634 90998 19	2,9*5*634 90998 19
8,75	2,94803 98915 50	2,9*5*803 98915 50
8.80	2,96647 93848 38	2,96647 93*9*48 38*
8.98	2,99666 28127 54	2,99666 *4*8127 54*
9,31	3,05122 92504 78	3,05122 92*6*04 *87*
9,51	3,08382 87889 22	3,08382 878*90* 22
14,1	3,75499 66666 56	3,75499 66*711 04*
14,3	3,78153 40798 26	3,78153 40*802 38*

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UNPUBLISHED MATHEMATICAL TABLES

159[A].—NATIONAL PHYSICAL LABORATORY (Great Britain), *Tables of Binomial Coefficients*. 20 quarto pages. Deposited with the ROYAL SOCIETY (no. 15).

The Binomial coefficients $C_n = \binom{x}{n}$ are given to six decimal places for $x = 0(.001)1$; $n = 2(1)8$. These are coefficients in the Newton-Gregory formula for interpolation with forward differences.

160[D,L].—INSTITUT DE MATHÉMATIQUES APPLIQUÉES, École Polytechnique de l'Université de Lausanne (Switzerland). Manuscript in possession of the author.

Table of the function $F(x) = \sin x/x$ and of its derivatives $d^nF(x)/dx^n$ for $x = 0(.01)4$, $n = 0(2)16$, 10D.

This table has been obtained by subtabulation from another table calculated on the base of series for $x = 0(.1)4.3$. A National accounting machine (class 3000) was used for the subtabulation. The error in the original tables does not exceed .55 units of the last order.

CH. BLANC

161[D].—SUBMARINE SIGNAL DIVISION OF THE RAYTHEON MANUFACTURING COMPANY, Transducer Department, Boston, *Sines and Cosines of the Decimal Circle*. 10 lithographed octavo pages of tables. Deposited with the ROYAL SOCIETY (no. 2).

This table gives $\sin \alpha$ and $\cos \alpha$ for $\alpha/2\pi = 0(.001)1$ to 5D when the entry is less than 1/6 and to 4D otherwise.

162[E].—A. YOUNG & T. MURPHY, "Tables of the Langevin Function and its inverse," 6 leaves. Deposited in UMT File, also in the depository of the ROYAL SOCIETY (no. 21).

The function $L(x) = \coth x - x^{-1}$ is tabulated for $x = 0(.01)7.50$ to 6D.

A comparison of this table with previously existing tables reveals errata in the previous tables which will be found in MTE **228**.

163[G].—E. M. IBRAHIM, *Tables for the plethysm of S-functions*. Deposited with the ROYAL SOCIETY (no. 1).

This comprises about two dozen large sheets of manuscript. The purpose of these specialized algebraic tables is described by the author in *Quar. Jn. Math.* (Oxford Series), Ser. 2, v. 3, 1952, p. 50–55. They relate to a special type of multiplication connected with symmetric functions and extend to partitions of a total degree of 18.

164[I,L].—C. W. JONES, J. C. P. MILLER, J. F. C. CONN, & R. C. PANKHURST, *Tables of Chebyshev Polynomials*. 2 typed double-foolscap sheets. Deposited with the ROYAL SOCIETY (no. 7).

These tables give exact values of $C_n(x) = 2\cos(n \cos^{-1} \frac{1}{2}x)$ for $x = 0(.02)2$ and $n = 1(1)10$. For $n = 8, 9, 10$ they extend and complete the curtailed values given in the table described in RMT **381**, *MTAC*, v. 2, p. 262. [See also RMT **1103**.]

165[L].—ADMIRALTY RESEARCH LABORATORY, *Solution of the Equation* $(y'')^2 = yy'$ *and two other Equations*. 3 foolscap manuscript pages of tables and 4 pages of description. Deposited with the ROYAL SOCIETY (no. 9).

The solution of the equation $(y'')^2 = yy'$ for which $y = 0$, $y' = 1$ when $x = 0$, is tabulated to 6 decimals for $x = 0(.05).5(.1)6$ and facilities are provided for interpolation.

The integral

$$2\pi^{-\frac{1}{2}}\beta^{-2}\exp(-\rho^2)\int_0^\beta \epsilon I_0(2\rho\epsilon)\exp(-\epsilon^2)d\epsilon$$

is tabulated to 4 decimals for the range $\beta = 0(.25)4$, $\rho = 0(.25)5$. Interpolation is possible in this table, but no differences are provided.

The root x of the equation $u \sin x - \cos x + e^{-ux} = 0$ that lies between π and 2π is tabulated to 4 figures for $u = .1(.01).3(.02)2$ and $u^{\frac{1}{2}} = 0(.02).5$. Interpolation is linear.

166[L].—M. S. Corrington, *Tables of Fresnel integrals, modified Fresnel integrals, the probability integral, and Dawson's integral.* Radio Corporation of America, R.C.A. Victor Division. 25 quarto pages. Deposited with the Royal Society (no. 4).

These tables give values for $x = \frac{1}{2}\pi u^2 = 0(.001).02(.01)2$ of the functions

$$C(u) = \tfrac{1}{2}\int_0^x J_{-\frac{1}{2}}(t)dt = \int_0^u \cos\left(\tfrac{1}{2}\pi t^2\right)dt$$

$$S(u) = \tfrac{1}{2}\int_0^x J_{\frac{1}{2}}(t)dt = \int_0^u \sin\left(\tfrac{1}{2}\pi t^2\right)dt$$

$$Ch(u) = \tfrac{1}{2}\int_0^x I_{-\frac{1}{2}}(t)dt = \int_0^u \cosh\left(\tfrac{1}{2}\pi t^2\right)dt$$

$$Sh(u) = \tfrac{1}{2}\int_0^x I_{\frac{1}{2}}(t)dt = \int_0^u \sinh\left(\tfrac{1}{2}\pi t^2\right)dt$$

$$H(x^{\frac{1}{2}}) = \frac{\sqrt{2}}{\pi}\int_0^x K_{\pm\frac{1}{2}}(t)dt = \int_0^x \frac{e^{-t}}{\sqrt{\pi t}}dt = \frac{2}{\sqrt{\pi}}\int_0^{x^{\frac{1}{2}}} e^{-t^2}\,dt$$

and

$$D(x^{\frac{1}{2}}) = \frac{\sqrt{2}}{i\pi}\int_0^{-x} K_{\pm\frac{1}{2}}(t)dt = \int_0^x \frac{e^{t}}{\sqrt{\pi t}}dt = \frac{2}{\sqrt{\pi}}\int_0^{x^{\frac{1}{2}}} e^{t^2}\,dt$$

Two versions of the tables are given, one to 5D and another to 8D, with an error up to 2 final units.

167[L].—C. Mack & M. Castle, *Tables of* $\int_0^a I_0(x)dx$ *and* $\int_a^\infty K_0(x)dx$. Deposited with the Royal Society (no. 6).

The integrals have been tabulated to 9D for the range of argument $a = 0(.02)2(.1)4$. A brief description of the method of computation is given and also of the extent to which the tables are interpolable.

168[L].—National Physical Laboratory, *Tables of the complex Jacobian zeta function*. 9 foolscap pages. Deposited with the Royal Society (no. 14).

The Jacobian Zeta function Z_n of modulus $k = \sin\alpha$ can be found from the tabulated function f_1 by the relation

$$Z_n(K\psi_1 + iK'\phi,k) = f_1(\psi_1,\phi,\alpha) + if_2(\psi_1,\phi,\alpha) - \tfrac{1}{2}i\pi\phi/K$$
$$f_2(\psi_1,\phi,\alpha) = f_1(1 - \psi_1, 1 - \phi, \tfrac{1}{2}\pi - \alpha)$$

where K, K' are the complete elliptic integrals of modulus k. f_1 to 3 significant figures is given for $\psi_1 = 0(.1)1$; $\phi = 0(.1)1$; $\alpha = 5°(5°)85°$. No provision is made for interpolation.

169[L].—National Physical Laboratory, *Integrals of Bessel Functions*. 2 quarto pages. Deposited with the Royal Society (no. 17).

10D values of $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$ for $x = 0(.5)50$. No provision is made for interpolation.

170[L].—National Physical Laboratory, *Table of*

$$\int_0^{2\pi} J_1^2\ (2k \sin \tfrac{1}{2}\theta) \cos^2 \tfrac{1}{2}\theta\, d\theta.$$

1 quarto page. Deposited with the Royal Society (no. 18).

4D values are given for $k = 0(.1)10$. The table is interpolable using second differences, but no differences are given.

AUTOMATIC COMPUTING MACHINERY

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Technical Developments

A SPECIAL PURPOSE DIGITAL COMPUTER

1. **Design considerations.** The design considerations of a special purpose computer for the solution of a large number of simultaneous linear, algebraic equations depend not only on the number of equations with which the computer must deal but also upon the properties of the matrix of the equations, the time to be allowed for computation and the required accuracy of the solution. In this particular case, it was estimated that up to 1200 equations might be expected and the arbitrary time of one day was allowed for computation, after the problem had been set up on the computer. An accuracy of one part in 100 was demanded in the solutions.

A set of simultaneous equations can be represented in the matrix notation as

$$AX = C \tag{1}$$

where A is the matrix of coefficients, a_{ij}, X is the column vector of unknowns x_i, and C is the column vector of constant terms on the right hand side of the equations, c_i. The first equation in the set of equations would look like this.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1. \tag{2}$$

There are many methods of solving such equations [1,2] but the most suitable, from the standpoint of machine computation, is that of Gauss-Seidel.[2] This procedure is an iterative procedure, each successive iteration yielding a better estimate of the desired vector X. Thus at the end of the first iteration one has $X^{(1)}$, at the end of the second, $X^{(2)}$, etc. One is assured that $X^{(n)}$ approaches X as n becomes very large if the equations are normal equations.

The computation associated with the Gauss-Seidel procedure is as follows:

$$\begin{aligned} x_i^{(p)} = (c_i - a_{i1}x_1^{(p)} - a_{i2}x_2^{(p)} \cdots - a_{i,i-1}x_{i-1} \\ - a_{i,i+1}x^{(p-1)} - \cdots a_{in}x_n^{(p-1)})/a_{ii} \end{aligned} \tag{3}$$

where $x_i^{(p)}$ is the i^{th} unknown being calculated in the p^{th} iteration. Equation (3) simply states that to compute $x_i^{(p)}$ one takes c_i, subtracts the products indicated, and divides the whole by a_{ii}. The products in eq. (3) consist of two kinds, first products involving $x_j^{(p)}$ and $x_k^{(p-1)}$. That is, as an unknown is calculated in the p^{th} iteration, this new value of the unknown is used in successive calculations in that iteration.

Two different methods of performing the computation indicated in eq. (3) suggest themselves. The difference lies in the method of storage. One may feed the coefficients, a_{ij}, into the arithmetic section of the machine according to rows and the unknowns, $x_i^{(p)}$, in synchronism with the coefficients and perform the desired operation. The diagonal terms, a_{ii}, must be left at the end of each row as they are the divisor of the accumulated sum. This method requires that each coefficient be included in order that proper synchronism be maintained, even though the coefficient be zero. The alternate to this method consists of feeding only non-vanishing matrix coefficients into the arithmetic section, as before, and storing the unknowns in such a fashion that they can be called from storage in an arbitrary order. This allows considerable saving in storing matrix coefficients, if the percentage of non-vanishing terms is small, at the expense of some complexity in the storage of the unknowns. In the problems we expected to deal with, only 1% of the coefficients were of the non-vanishing variety and the second method outlined above was selected. Fig. 1 is a block diagram of the computer using this method.

2. Storage. For a given problem, the order of the matrix coefficients entering the computation is fixed and furthermore is periodic, with period of one iteration. Magnetic tape was selected as the medium for storing these terms, a continuous loop of tape being used. The matrix coefficients are recorded serially along one channel of the multi-channel tape, along with an order that specifies the type of term, i.e., c_i, a_{ij} or a_{ii}. In an adjacent channel of the tape is recorded the column address of the matrix coefficient. It is this address that is used to specify the location of the corresponding unknown, x_i. The coefficient-order, called a word, is stored in 25 binary digits. A front view of the computer is given in the frontispiece. This shows the tape trans-

port mechanism with a continuous loop of tape. The tape is run at 15 in./sec. and the coefficients are read off at 20 words per second.

Since the unknowns do not appear in the computation in a systematic fashion, it is necessary to provide random access to this storage, with the access time commensurate with the speed at which matrix coefficients are produced. A magnetic drum was selected for this storage. The drum revolves at approximately 59 rps and thus maximum access time to any word on the drum is roughly 17 milliseconds. The capacity of the drum storage is dictated by the maximum number of equations to be handled; in this case one must allow for storage of 1200 words. Fig. 2 is a photograph of the magnetic drum.

Words on the drum are stored serially, 64 words of 25 binary digits in each of the 19 tracks around the periphery of the drum. A "return to zero" system of pulse recording is used and pulses are recorded at a density of about 60 pulses per inch. The desired track on the drum is obtained by switching a matrix of gates. This switching is done by inserting the track address into an address register, AR. The desired angular position on the

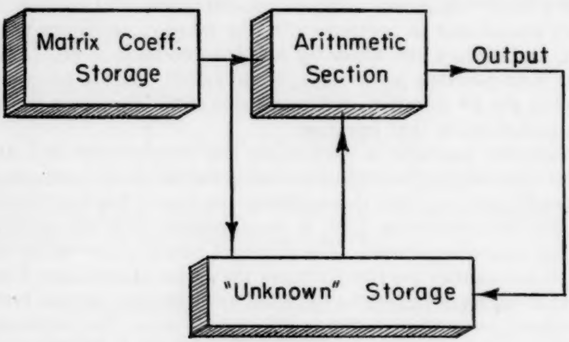

FIG. 1. Block Diagram of Computer

drum is obtained by counting the number of words that have passed under a head, and noting coincidence between this count and the angular position as specified by the AR. When coincidence occurs, the operation (read or write) may proceed. This BOW (Beginning of Word) counter is set to zero once every revolution by an origin pulse.

3. Arithmetic section. All computation is performed in the arithmetic section of the computer. The following operations and the time required for each are indicated below.

Addition	200 microseconds
Subtraction	200 microseconds
Multiplication	8 milliseconds
Division	8 milliseconds

The operation times listed above obviously do not include 17 milliseconds (maximum) required to secure one of the operands from the drum storage.

All computation is performed in parallel,[3] that is the four operations

above are performed as parallel transfers from register to register. Carries proceed by means of delay circuits between digits. Thus for an n digit register, nd seconds must be allowed for carries to occur, where d is the delay time between digits. No time advantage over serial operation is obtained but it is possible to modify this method of operation to include a simultaneous carry[3] when faster storage becomes practical.

There are three registers in the arithmetic section, a multiplier register, MR, multiplicand register, MD, and a product accumulator, PA. The following transfers are possible: $\pm$ MR to PA, $\pm$ MD to PA. Of the 25 binary digits in the MR, MD registers, 20 digits are occupied by numbers and 5 digits by order. The binary point may be located either to the right of the least significant digit, or to the left of the most significant digit. The PA

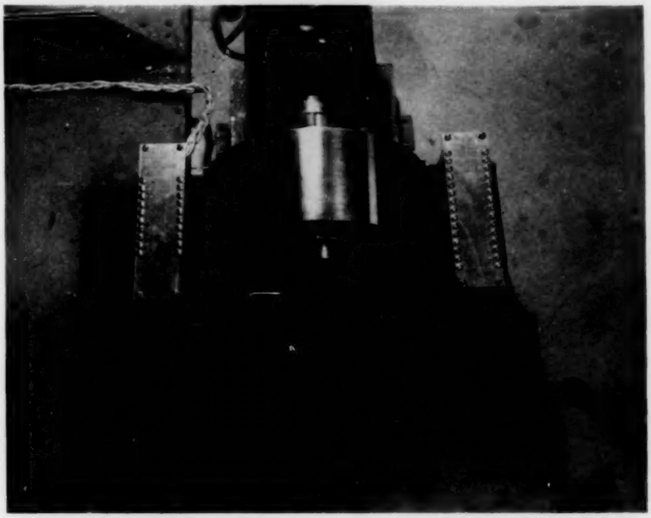

FIG. 2. Magnetic Drum Storage Unit

register contains 40 binary digits and the binary point is located at the center of the register. Each register may be used as an accumulator or as a shifting register.

In normal computation the matrix coefficients are read from the magnetic tape into the MR register, the address of the coefficient into AR, the corresponding unknown is read into the MD and the indicated operation is performed. In the case of division, the quotient $x_i^{(p)}$ is formed in the MD and is compared, digit by digit, with the number previously in the MD, i.e. $x_i^{(p-1)}$. Thus too great a difference in the two values indicates an error in computation (and the new value of the unknown is rejected) and very minor differences for all the unknowns may serve as a criterion for convergence of the solutions.

The frontispiece is a photograph of the complete computer. The rack on the left contains the power supply. The two racks adjacent to the power

supply carry the three registers of the arithmetic section and associated controls. The next rack from the left carries the address register, AR, and drum controls. The tape rack on the right has already been discussed.

4. Logical design. The computer is a purely binary machine. Numbers are represented in the binary notation in order to take advantage of the simplicity of arithmetic operations with this notation.[4] Negative numbers are represented as the "one's" complement of the absolute value of the number. That is, if the number 10 is written as 0000001010, the number -10 is written as 1111110101. In other words, the complement of the absolute value of the number has been taken with respect to the modulus of the register minus one, $M - 1$. With this representation of negative numbers, the product of two numbers may be formed without taking cognizance of the signs of the numbers as such. Let $-A$ be represented as $C(A)$, the complement of A. Then

$$C(A) \cdot B = C(AB)$$

and

$$C(A) \cdot C(B) = AB$$

if the partial products are accumulated modulo $M - 1$. The modulus of an accumulator may be reduced by 1 by the addition of "end around carry." It consists of feeding a carry from the last digit of the register back to the first digit. It is easy to show that the modulus of a n digit register, modulus 2^n, is reduced to a modulus of $2^n - 1$, with the addition of end around carry.

5. Logical elements. The logical elements or components of the computer were designed on the basis of obtaining reliable operation over wide variations of supply voltages and tube characteristics. The components were built

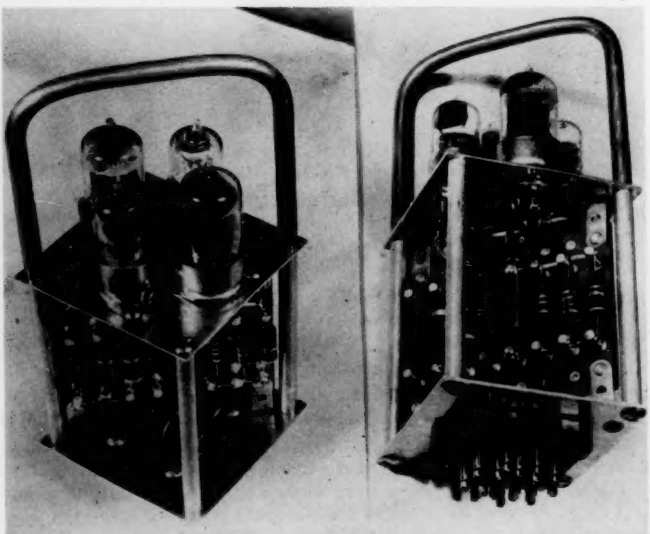

FIG. 3. Typical Plug-in Unit

up on plug-in chassis, as shown in Fig. 3, for ease of maintenance of the computer. The components include flip-flops, delay circuits, gates, and cathode followers. Each element requires a 15–20 volt positive pulse input and its output may be of either polarity. The elements are insensitive to negative pulse inputs. Around 400 of these elements are used in the computer.

6. Operation. The computer has been in operation about 3 months and two problems have been set up and solved. One consisted of 73 equations with a computation time of 3 minutes per iteration. The other consisted of 793 equations and 20 minutes of computing time were required per iteration. In both cases, 15 iterations were sufficient to yield solutions of the desired precision.

J. P. WALKER, JR.

Haller, Raymond and Brown, Inc.
State College, Pa.

[1] FRAZER, DUNCAN & COLLAR, *Elementary Matrices*. Cambridge, 1946.
[2] WHITTAKER & ROBINSON, *The Calculus of Observations*. London, 1924.
[3] ENGINEERING RESEARCH ASSOCIATES, *High Speed Computing Devices*. McGraw Hill, 1950.
[4] INSTITUTE FOR ADVANCED STUDY, *Preliminary Discussion of the Logical Design of an Electronic Computing Instrument*. Princeton, 1947.

BIBLIOGRAPHY OF CODING PROCEDURE

The material described below is among that which has been added to the collection at the National Bureau of Standards Computation Laboratory. An equivalent collection is available at the National Bureau of Standards Institute for Numerical Analysis.

Material for inclusion in these collections should be sent to the Computation Laboratory, National Bureau of Standards, Washington 25, D. C., for the attention of J. H. WEGSTEIN.

16. Remington Rand, *Prefab Tab*. A routine is described which generates a routine for turning UNIVAC into the logical equivalent of a fictitious punch card tabulator.
17. Remington Rand, *Preliminary Exposition, UNIVAC Short Code*.
18. Willow Run Research Center, University of Michigan, *Willow Run Research Center Memoranda*. The following subjects are included:
    MIDAC characteristics
    A list of problems and machines with which they were solved
    Programming on the Whirlwind Computer
    Indoctrination of potential coders
    A suggested method of processing problems on MIDAC
    MIDAC Logic: word composition
    MIDAC magnetic drum control equipment
    Preliminary coding manual for MIDAC
19. Analysis Laboratory, Calif. Inst. of Technology, Bibliography of articles on computing machines published prior to June, 1949.
20. Digital Computer Laboratory, Mass. Inst. of Technology, *A Short Guide to Coding*, May, 1951.
21. Digital Computer Laboratory, Mass. Inst. of Technology, *Short Guide to Coding and Whirlwind I Operation Code*.

22. Digital Computer Laboratory, Mass. Inst. of Technology, ***Programming for Whirlwind I.*** This report tells what Whirlwind I is and what it can do. It presents essentials and examples of Whirlwind I programming and suggests various programming techniques. It contains an appendix which discusses handling numbers in the machine and the operation code.

23. National Bureau of Standards Report 1612, April 21, 1952.
SEAC Operating and Programming Notes, I.
1. Operation of SEAC Control Panel as of October, 1951
2. Non-mathematical Routines
3. Conversion Subroutines for Integers
4. Decimal to Binary and Binary to Decimal Subroutines
5. Subroutine for 2^x and e^x
6. Subroutines for $\sqrt{N}$, $\sqrt[3]{N}$, $\sqrt[n]{N}$
7. Subroutine for Sin X and Cos X

24. National Bureau of Standards Report 1807, July 22, 1952.
SEAC Operating and Programming Notes, II.
8. Subroutine for $\tan^{-1} a/b$
9. Hexadecimal-decimal Converter Table
10. Four-Hexi Converter Table
12. Subroutine for Decimal to Binary Conversion of a Double Precision Number with Fixed Binary Point
13. Subroutine for i-th Differences, $i=1, 2, \ldots, 6$
14. Three-address System and Base Order
15. Preparation of Pure Hexadecimal Coding for Punched-Card Composition

25. National Bureau of Standards Report 1873, August 22, 1952.
SEAC Operating and Programming Notes, III.
16. Magnetic Tape to Wire Transfer and Check Routine
17. Subroutines for Basic Arithmetic Operations and Square Root in Floating Decimal Form
18. General Dump Routine (512 or 1024 Word Memory)
19. Memory Decomposition Routine (512 or 1024 Words)
20. Flow Charts for Magnetic Tape Checking Routines

26. National Bureau of Standards Report 1917, September 12, 1952.
SEAC Operating and Programming Notes, IV.
21. Composition Routine (SEBBE)
22. Basic Arithmetic Operations I, Floating Binary Point, Single Precision Numbers
23. Subroutine for Cube Root: Single Precision, Floating Binary Point
24. Subroutine for sin z, cos z, $z = x + iy$.
25. Corrections and Changes to Technical Memoranda Nos. 1–24.

27. RUTH B. HORGAN, *SWAC Coding Manual*, National Bureau of Standards Report 2047, November 4, 1952, 37 p.

This coding guide gives a good introduction to the art of coding for the SWAC, although it is incomplete in certain respects. For example, the relative time to obey each command is not given so it is not possible to minimize or even estimate the time required to do a calculation with the SWAC. The number of binary digits in a word is not stated explicitly, nor whether the modulus or complement convention is used.

In Chapters 2 and 3 the input-output procedures and possibilities are described in detail. Chapter 4 is devoted to a method of verifying the correctness of the store and the coding precautions which have to be taken to enable this to be done readily. The following two chapters describe how commands can be modified and tallies kept.

The last chapter describes the use of sub-routines and the necessary conventions although it is not actually stated how a sub-routine is entered from a program.

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BIBLIOGRAPHY Z

1038. ALBERT A. GIRLACH, "Test pulse generator for digital computers," *Electronics*, v. 25, Nov. 1952, p. 158–160.

This paper gives a description of a pulse generator which was built using multivibrator and gate circuits. It can be triggered either internally or externally. The outputs are a clock output plus two pulse outputs and their complements. The width and amplitude of each output can be varied independently. A block diagram and complete circuit diagrams are given.

ERNEST F. AINSWORTH

NBSEC

1039. S. E. GLUCK, "The Electronic Discrete Variable Computer," *Electrical Engineering*, v. 72, Feb. 1953, p. 159–162.

A careful description of the EDVAC is given by the author. This machine has been operating successfully since the Spring of 1952 at the Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

The reader familiar with the SEAC will find much similarity between the two machines.

IDA RHODES

NBS

1040. INTERNATIONAL BUSINESS MACHINES CORPORATION, *Industrial Computation Seminar*, edited by Cuthbert C. Hurd, Dec. 1949, 173 pages. 22 × 28.5 cm.

This is one of a series of seminars that was held for research engineers and scientists who have active interests in the mathematical solution of physical problems by the application of punched card techniques to the computing art. The following articles were discussed:

1. "The future of high-speed computing," by JOHN VON NEUMANN.
2. "Some methods of solving hyperbolic and parabolic partial differential equations," by RICHARD W. HAMMING.
3. "Numerical solution of partial differential equations," by EVERETT C. YOWELL.
4. "An eigenvalue problem of the Laplace operator," by HARRY H. HUMMEL.

5. "A numerical solution for systems of linear differential equations occurring in problems of structures," by Paul E. Bisch.
6. "Matrix methods," by Kaiser S. Kunz.
7. "Inversion of an alternant matrix," by Bonalyn A. Luckey.
8. "Matrix multiplication on the IBM card-programmed electronic calculator," by John P. Kelly.
9. "Machine methods for finding characteristic roots of a matrix," by Franz L. Alt.
10. "Solution of simultaneous linear algebraic equations using the IBM Type 604 Electronic Calculating Punch," by John Lowe.
11. "Rational approximation in high-speed computing," by Cecil Hastings, Jr.
12. "The construction of tables," by Paul Herget.
13. "A description of several optimum interval tables," by Stuart L. Crossman.
14. "Table interpolation employing the IBM Type 604 Electronic Calculating Punch," by Everett Kimball, Jr.
15. "An algorithm for fitting a polynomial through n given points," by F. N. Frenkiel and H. Polachek.
16. "The Monte Carlo Method and its application," by Mark Kac and M. D. Donsker.
17. "A punched card application of the Monte Carlo Method," by P. C. Johnson and F. C. Uffelman.
18. "A Monte Carlo Method of solving Laplace's equation," by Everett C. Yowell.
19. "Further remarks on stochastic methods in quantum mechanics," by Gilbert W. King.
20. "Standard methods of analyzing data," by John W. Tukey.
21. "The application of machine methods to analysis of variance and multiple regression," by Robert J. Monroe.
2. "Examples of enumeration statistics," by W. Wayne Coulter.
23. "Transforming theodolite data," by Henry Schutzberger.
24. "Minimum volume calculations with many operations on the IBM Type 604 Electronic Calculating Punch," by William D. Bell.
25. "Transition from problem to card program," by Gregory J. Toben.
26. "Best starting values for an iterative process of taking roots," by Preston C. Hammer.
27. "Improvement in the convergence of methods of successive approximations," by L. Richard Turner.
28. "Single order reduction of a complex matrix," by Randall E. Porter.
29. "Simplification of statistical computations as adapted to a punched card service bureau," by W. T. Southworth and J. E. Bachelder.
30. "Forms of analysis for either measurement or enumeration data amenable to machine methods," by A. E. Brandt.
31. "Remarks on the IBM Relay Calculator," by M. Lotkin.
32. "An improved punched card method for crystal structure factor calculations," by M. D. Grems.

33. "The calculation of complex hypergeometric functions with the IBM Type 602-A Calculating Punch," by HARVEY GELLMAN.

34. "The calculation of roots of complex polynomials using the IBM Type 602-A Calculating Punch," by JOHN LOWE.

35. "Practical inversion of matrices of high order," by WILLIAM D. GUTSHALL.

Several of the papers that were discussed dealt with specific problems as encountered in actual practice and their particular solution and are useful only in that they illustrate the type of approach that the authors used. Some of the other articles, such as no. 3 by E. C. Yowell, describe methods which are rather general in approach and could be applied to a large number of problems involving solution of partial differential equations. The article by Donsker and Kac wherein they discuss the Monte Carlo Method and its application is also rather general and so could be applied to numerous problems. Finally the articles by Kunz on matrix methods and Tukey on the standard methods of analyzing data are more theoretical and are excellent summaries but contain no mention of any direct application to punch card methods.

H. BREMER

NBSCL

1041. HERBERT KOPPEL, "Digital computer plays NIM," *Electronics*, v. 25, Nov. 1952, p. 155–157.

Engineers of the W. L. Maxon Corporation have constructed an automatic machine for playing NIM with a human opponent. It can be adjusted so that the human can win if he plays a perfect game or so that he can never win. Digital computer techniques such as gates, binary counters, etc., were used in its construction. The method of operation and a block diagram are given but no circuitry details are shown.

ERNEST F. AINSWORTH

NBSEC

1042. W. H. MACWILLIAMS, JR., "Computers—past, present, and future," *Electrical Engineering*, v. 72, Feb. 1953, p. 116–121.

The author gives an exceedingly clear, yet non-technical, exposition of many computing devices, both analogue and digital, which humanity has been utilizing for the past three centuries. This informative article should appeal to every member of that rapidly growing circle of engineers and mathematicians, who—either as constructors or users—are closely connected with the art of computation.

The reviewer is disappointed that the author's glimpse into the future failed to perceive a much needed system of electronic sorter-collators, whose lack is felt by agencies handling huge masses of data.

IDA RHODES

NBSCL

1043. Office of Naval Research, *Digital Computer Newsletter*, v. 5, January 1953, 8 p.

The contents are as follows:

1. Naval Proving Ground Calculators
2. Whirlwind I
3. Moore School Automatic Computer
4. The SWAC
5. Aberdeen Proving Ground Computers:
   The ORDVAC
   The EDVAC
   The ENIAC
   The BELL
   IBM-CPC
6. The Circle Computer
7. The Jacobs Instrument Company Computer (JAINCOMP)
8. The ELECOM Computers
9. University of Illinois Computer (ILLIAC)
10. Hughes Aircraft Company Computer
    Data Processing and Conversion Equipment
1. Flying Typewriter
2. The Charactron

List of Computing Services

1044. Office of Naval Research, *Digital Computer Newsletter*, v. 5, April 1953, 12 p.

The contents are as follows:

1. Naval Proving Ground Calculators
2. Whirlwind I
3. Computer Research Corporation Computers
   CADAC 102-A
   CRC 105
   CRC 107
4. Moore School Automatic Computer (MSAC)
5. Air Force Missile Test Center Computer (FLAC)
6. The SEAC
7. The IAS Computer
8. The SWAC
9. The MONROBOT
10. The Circle Computer
11. The Jacobs Instrument Company Computers (JAINCOMPS)
12. Consolidated Electronic Digital Computer Model 30–201
13. The ERA 1103 Computer
14. The Rand Corporation Computer
15. Aberdeen Proving Grounds Computers
    Data Processing and Conversion Equipment
1. TELEDUCER
2. SADIC

3. Benson-Lehner Incremental Plotter
4. Coleman Digitizer
5. Ferro-Resonant Flip-Flop
6. Logrinc Automatic Graph Followers

List of Computing Services

Computer and Numerical Analysis Courses

1. Massachusetts Institute of Technology
2. Computer Research Corporation

1045. L. PACKER & W. J. WRAY, JR., "Germanium photo-diodes read computer tapes," *Electronics*, v. 25, Nov. 1952, p. 150–151.

This article describes a system for reading Teletype paper tapes using 1N77 photo-diodes. Complete circuit details are given for the amplifier and shaper. It operates over a range from one to 35,000 pulses per second. Mechanical details of the tape transport are not discussed.

ERNEST F. AINSWORTH

NBSEC

1046. UNIVERSITY OF SIDNEY, ***Proceedings of Conference on Automatic Computing Machines***, held in the Department of Electrical Engineering, University of Sidney, Australia, in conjunction with the Commonwealth Scientific and Industrial Research Organization, Aug. 1951, 220 pages. 20 × 24.8 cm.

The first of two sessions contained "An introduction to automatic calculating machines" and a paper entitled "Automatic digital calculating machines" both by D. R. HARTREE. Two other papers discussed the C.S.I.R.O. (Commonwealth Scientific and Industrial Research Organization) Differential Analyser and the C.S.I.R.O. Radiophysics MK. I Automatic Computer. The latter is an acoustic-delay-line-memory digital computer with magnetic drum and punched card input-output. In the discussion of the use of superfluous binary digits for error-detecting which followed, Hartree considered that "the use of error detecting procedures at each operation of the machine was a counsel of despair." He felt that experience has shown that computers do not go wrong often enough to warrant this.

In the second session, Hartree gave an introduction to programming using the EDSAC for illustration; also he presented a paper on numerical methods used with automatic calculating machines. The latter included a summary of various methods of determining roots of polynomial equations with automatic machines. T. PEARCEY presented papers on programming for the C.S.I.R.O. digital machine and for punched-card machines and on the functional design of an automatic computer. Some of the other papers presented were: (1) "Some analogue computing devices," (2) "Digital-analogue conversions," (3) "An analogue computer to solve polynomial equations with real coefficients," and (4) "Some new developments in equipment for high-speed digital machines." The last paper dealt with a high-speed magnetic switching device, a single electron tube scale-of-ten numeral-displaying counter, a single tube which combines the functional operations

of a bi-stable element and gate, giving several applications and details of a magnetic-drum digital storage system.

J. H. WEGSTEIN

NBSCL

1047. B. G. WELBY, "Intermittent-feed computer tape reader," *Electronics*, v. 26, Feb. 1953, p. 115–117.

A tape reader is described which will operate at speeds up to 200 characters per second with teletype tape. Each character consists of as many as seven holes punched in the paper tape. Reading is accomplished by a photoelectric system, and no reading pins or contacts are required. The tape is friction driven by rollers which can be controlled intermittently. The maximum reading speed under intermittent control is not stated but appears to be about 100 characters per second. Reading causes no visible wear to the tape after 10,000 passes.

J. L. PIKE

NBSCL

NEWS

AIEE-IRE-ACM.—The joint AIEE-IRE-ACM Computer Conference Committee sponsored a Western Computer Conference at Los Angeles, California, on February 4, 5, and 6. The program for the meeting was as follows:

Feb. 4, 1953, 8:30 a.m.

Registration	Ballroom Floor

10:30–11:30 a.m.

Opening ceremonies	P. L. MORTON, *Chairman*, Univ. of Calif., Berkeley
Keynote addresses:	
The impact of computer development on the training and utilization of engineers	S. RAMO, Vice-President for Operations, Hughes Aircraft Co., Culver City, Calif.
Factors influencing the effective use of computers	R. D. HUNTOON, Chief, Corona Laboratories, NBS Corona, Calif.
Lunch	G. D. MCCANN, *Toastmaster*, Calif. Inst. of Tech., Pasadena
The scientific manpower problem	L. A. DUBRIDGE, Pres., Calif. Inst. of Tech., Pasadena

2:00–4:30 p.m.

Session I, Commercial applications	E. C. NELSON. *Chairman*, Hughes Aircraft Co., Culver City, Calif.
Commerical applications—The implications of Census experience	J. L. MCPHERSON, Bureau of the Census
Payroll accounting with ELECOM 120	R. F. SHAW, Electronic Computer Division, Underwood Corp.
Automatic data-processing in larger manufacturing plants	M. W. SALVESON and R. G. CANNING, Univ. of Calif., Los Angeles
The data-processing requirements of the Social Security problem	E. E. STICKELL, Bureau of Old-Age and Survivors Insurance
The processing of information-containing documents	G. W. BROWN and L. N. RIDENOUR, International Telemeter Corp.

Feb. 5, 1953, 9:00–11:30 a.m.

Session II, Applications to aircraft and missile design	C. STRANG, *Chairman*, Douglas Aircraft Co., Santa Monica, Calif.
Landing gear simulation using a differential analyzer	D. W. DRAKE, Lockheed Aircraft Corp.
The equivalent circuits of shells used in airframe construction	R. H. MACNEAL, Calif. Inst. of Tech.
Analog-digital techniques in autopilot design	R. L. JOHNSON and W. T. HUNTER, Douglas Aircraft Co.
Applications of computers to aircraft dynamics problems	D. DILL, B. HALL, R. RUTHRAUFF, Douglas Aircraft Co.

12:00–1:30 p.m.

Lunch	R. G. CANNING, *Toastmaster*, Univ. of Calif., Los Angeles
Luncheon Address:	
New equations for management	J. E. HOBSON, Director, Stanford Research Institute, Palo Alto

2:00–4:30 p.m.

Session III, Panel discussion	
An evaluation of analog and digital computers	G. D. MCCANN, *Moderator*, Calif. Inst. of Tech.

Panel: J. L. BARNES, Assoc. Director of Electro-Mechanical Engineering Dept., North American Aviation, Inc.
L. RIDENOUR, Vice-President, International Telemeter Corp.
F. STEELE, Vice-President in charge of Engineering Digital Control Systems, La Jolla, Calif.
A. W. VANCE, Research Section Head, RCA Laboratory, Princeton, N. J.

Feb. 6, 1953, 9:00–11:30 a.m.

Session IV, New developments in digital computer equipment	H. D. HUSKEY, *Chairman*, Wayne Univ.
The snapping dipoles of ferro-electrics as a memory element for digital computers	C. F. PULVARI, The Catholic Univ. of America
Magnetic reproducer and printer	J. C. SIMS, JR., Eckert-Mauchly Div., Remington Rand, Inc.
An improved cathode ray tube storage system	R. THORENSEN, NBSINA
Nonlinear resistors in logical switching circuits	F. A. SCHWERTZ, and R. T. STEINBACK, Mellon Institute

1:30–4:00 p.m.

Session V, New developments in analog computing equipment	C. H. WILTS, *Chairman*, Calif., Inst. of Tech., Pasadena
New laboratory for a three dimensional guided missile analysis	L. BAUER, Head, Project Cyclone, Reeves Instrument Corp.
A new concept in analog computers	L. CAHN, Chief Computer Engineer, Special Products Dept., Beckman Instruments, Inc.

A magnetically coupled low-cost high-speed shaft position digitizer	A. J. WINTER, Research Lab., Supervisor, Telecomputing Corp.
Solution of partial differential equations by difference methods using the electronic differential analyzer	R. M. HOWE, Univ. of Michigan
A syncro-operated differential analyzer	A. NORDSIECK, Univ. of Illinois

Following is a brief description of some of the main points brought out in the session on commercial applications. Mr. McPherson's talk outlined conclusions drawn from Census experience with the UNIVAC concerning the commercial applicability of the electronic digital computers. It is possible for them to exceed in efficiency any other available tool for many commercial purposes. They are not commercially ideal because their arithmetic power exceeds their input-output power. It is hoped that these machines will evolve into a powerful aid to business problems similar to the evolution of punched card equipment. The ELECOM 120 described by Mr. Shaw, can process the weekly payroll for 4,000 to 5,000 employees at the rate of about 30 seconds per employee including typing of checks and statements. It is a moderately priced decimal computer. In the discussion by Messrs. Salveson and Canning, it is mentioned that the ONR is sponsoring a project on "Production planning and scheduling" which includes the systems design specifications. Mr. Stickell discusses the possibilities and limitations of card-to-tape and tape-to-card equipment for sorting data in electronic media and the present lack of equipment for high-speed random access to data stored outside of a machine. In the last talk of Session I, Messrs. Brown and Ridenour state that more than 10,000 documents of all sorts can be processed per hour by a single machine which makes no demands on the size, shape, thickness, flatness, or degree of preservation of the documents being handled.

In Session II, "Applications to aircraft and missile design," Mr. Drake states that the computer simulation includes velocity square orifice damping, a nonlinear oleo air spring, wheel spin-up, slop in gear joints, and a broad range of airplane weights, effective wing lift, and landing speeds. The study suggests that this method of investigating actual landing gears is practical. Mr. MacNeal describes the development of analog computer techniques that should be helpful in solving practical problems in connection with aircraft fuselage and wing design. In the third paper of this session the roles assigned to digital and analog computation connected with autopilot design at Douglas are discussed and the associated reasons for such assignment are given. Messrs. Dill, Hall, and Ruthrauff discuss the basic methods for the solution of aircraft dynamics problems on both analog and digital computers; in this connection actual problems successfully solved on the computing equipment used by the Douglas Aircraft Company are outlined.

Mr. Pulvari begins Session IV with a discussion of "The snapping dipoles of ferroelectrics as a memory element of digital computers." The following description of his talk is quoted from the program: A sensitive pulse method has been developed for obtaining static remanent polarization data for ferroelectric materials. This method has been applied to study the effect of pulse length and amplitude, and decay of polarization on ferro-electric ceramic materials with fairly large crystalline orientation. Attempts have been made to develop electrostatically-induced memory devices using ferroelectric substances as a medium for storing information, particularly delay line, matrix, and tape type of memory devices; also a simple counting circuit using ferroelectric condensers as a bistable element has been designed. The paper by J. C. Sims, Jr., describes a new process for producing printed copy by magnetic fields which can be adapted to computer output printing. After the data are recorded on a magnetized surface, the latent magnet image is developed with a magnetic ink and transferred to paper and fixed. In the paper by R. Thorensen, it is shown that by altering somewhat the mode of operation of the cathode ray tube and by changing the associated gating circuitry, a system is obtained which operates successfully even under severe spillover signal distortion. Messrs. Schwertz and Steinback show how nonlinear resistors which are made by applying printed circuit techniques to such materials as standard plastic may be used to replace whole arrays of crystal rectifiers in certain logical switching circuits.

Some new developments in analog computing equipment were described in Session V. A new simulation laboratory constructed at Project Cyclone was planned to be large enough to permit the solution of complex three-dimensional guided missile problems. Mr. Bauer stated that it may also be used for other types of problems. In the paper "A new concept in analog computers" circuits of a coordinated low-cost analog computer are described and evaluated in terms of computing ability.

Symposium on automatic digital computation.—A symposium on Automatic Digital Computation was held at the National Physical Laboratory, Teddington, England on the 25th, 26th, 27th and 28th of March, 1953.

About 200 delegates attended the Symposium and of these about twenty came from other European countries and five from the U.S.A. The United States delegates were GERTRUDE BLANCH and R. J. SLUTZ, NBS; R. HULSIZER, University of Illinois; J. C. McPHERSON, I.B.M. World Headquarters, New York; and S. KAUFMAN, Shell Development Company.

The computing machine developed at the National Physical Laboratory, the ACE Pilot Model, was demonstrated at various times while the Symposium was in progress.

The programme for the Symposium was as follows:

March 25, 1953, 11:30 a. m.–1:00 p.m.

Session 1	E. T. GOODWIN, *Chairman*, NPL
Opening remarks	E. C. BULLARD, Director, NPL
Address	D. R. HARTREE, Cambridge Univ.

2:00 p.m. to 5:00 p.m.

Session 2, British Machines	F. M. COLEBROOK, *Chairman*, NPL
ACE Pilot Model	J. H. WILKINSON, NPL
EDSAC	M. V. WILKES, Cambridge Univ.
Operating and engineering experience gained with LEO	J. M. M. PINKERTON, Messrs. J. LYONS and Co.
MADAM	F. C. WILLIAMS, Manchester Univ.
MOSAIC, the Ministry of Supply Automatic Computer	A. W. M. COOMBS, Post Office Research Station
NICHOLAS	N. D. HILL, ELLIOTT Bros.
Advance notes on RASCAL	E. J. PETHERICK, RAE
The T. R. E. High-Speed Digital Computer	A. M. UTTLEY, TRE

March 26, 1953, 9:30 a.m.–1:00 p.m.

Session 3, Programming	E. T. GOODWIN, *Chairman*, NPL
Optimum coding	G. G. ALWAY, NPL
Micro-programming and the choice of order code	J. G. STRINGER, Cambridge Univ.
Conversion routines	E. N. MUTCH and S. GILL, Cambridge Univ.
Getting programmes right	S. GILL, Cambridge Univ.

2:00–5:00 p. m.

Session 4, Design	J. H. WILKINSON, *Chairman*, NPL
Special requirements for commercial or administrative applications	T. R. THOMPSON, Messrs. J. LYONS and Co.
Input and output	D. W. DAVIES, NPL
Echelon storage systems	D. O. CLAYDEN, NPL
Serial digital adders for a variable radix of notation	R. TOWNSEND, British Tabulating Machine Co.

March 27, 1953, 9:30 a.m.–1:00 p.m.

Session 5A, The Utilization of Computing Machines—I	E. T. Goodwin, *Chairman*, NPL
Mathematics and computing	A. van Wijngaarden, Mathematical Centre, Amsterdam
Linear algebra on the Pilot ACE	J. H. Wilkinson, NPL
The numerical solution of ordinary differential equations	L. Fox and H. H. Robertson, NPL
The solution of partial differential equations by automatic calculating machines	N. E. Hoskin, Manchester University

2:00 p.m.–5:00 p.m.

Session 6A, The Utilization of Computing Machines—II. Mathematical tables	L. Fox, *Chairman*, NPL E. T. Goodwin, NPL
Applications of electronic machines in pure mathematics	J. C. P. Miller, Cambridge University
'Monte Carlo' methods for the iteration of linear operators	J. H. Curtiss, NBS (presented by G. Blanch)

9:30 a.m.–1.00 p.m.

Session 5B, Circuitry and Hardware	E. A. Newman, *Chairman*, NPL
Gates and trigger circuits	W. W. Chandler, Post Office Research Station
Mercury delay line storage	M. Wright, NPL
Applications of magnetostriction delay lines	R. C. Robbins and R. Millership, Messrs. Elliott Bros.
Cathode ray tube storage	T.Kilburn, Manchester Univ.
Memory studies at the National Bureau of Standards, Washington, D. C., U.S.A.	R. J. Slutz, NBS

2:00 p.m.–5:00 p.m.

Session 6B, Servicing and Maintenance	F. M. Colebrook, *Chairman*, NPL
Preventive or curative maintenance	E. A. Newman, NPL
Experience with marginal checking and automatic routining of the EDSAC	M. V. Wilkes, M. Phister, Jr. and S. A. Barten, Cambridge Univ.
Diagnostic programmes	R. L. Grimsdale, Ferranti Ltd.
Component reliability in the computing machine at Manchester University	A. A. Robinson, Ferranti Ltd.

March 28, 1953, 9:30 a.m.–11:00 a.m.

Session 7, Medium-Size Digital Computing Machines	J. R. Womersley, NPL
The Harwell Computer	E. H. Cooke-Yarborough, AERE
The A.P.E.(X)C. A low cost electronic calculator	A. D. Booth, Birkbeck College
The Elliott-N.R.D.C. Computer 401—A demonstration of computer engineering by packaged unit construction	W. S. Elliott, H. G. Carpenter and A. St. Johnston, Elliott Bros.

OTHER AIDS TO COMPUTATION

COMPARISON OF AMERICAN ELECTRIC DESK CALCULATORS

The following list of advantages and disadvantages of the three American electric desk calculators is intended to be an impartial survey of non-debatable facts concerning the use of the machines for statistical and technical problems.

Anything common to all three machines—on either side of the ledger—is excluded. To save space, each item is entered for the odd machine, that is, an advantage for two of them is entered as a disadvantage of the third.

No attempt is made to assess ruggedness, service, or length of life.

All three machines have features which, though unique, do not clearly belong on one side of the ledger or the other. Examples:

1. Monroe claims "velvet touch," which refers to all the keys having the same pressure to operate.
2. Marchant features higher cycling speed and continuous mesh mechanism in driving the counter wheels. These two almost exactly compensate each other.
3. Friden offers the possibility of adjusting each wheel of the accumulating counter individually.

The comparison is made between the latest models as of the end of 1952, and for each the most complete machine, for which the current prices are $775 for Monroe and Friden and $815 for Marchant. The Marchant has complete carry-over on all models; the Monroe has it as a standard item only on the full size (automatic) machine; for the Friden it is not standard on any model. It should be pointed out that if the work to be done does not call for a full size machine, the prices differ sharply. In all cases, it pays to examine the range of models of all three before purchasing. No one of the machines stands out for all-purpose operation.

No comparison is made between these machines and other calculators, such as the Olivetti and Remington Rand printing calculators, the Curta 8-ounce hand operated machine, and the Swedish Facit (the latter formerly made here by R. C. Allen).

Both Monroe and Marchant offer octal calculators in their line. Monroe and Friden are about to bring forth electronic machines in team with IBM. Friden has a square root model available ($1300).

It is not possible, without exhaustive research, to evaluate the machines in terms of hourly production of written answers. It is probably true that for a variety of work (typical in technical applications) the difference in production speed is slight.

Friden advantages:

1. Zero keys can be raised to lock (positively) any single column.
2. Rounding (reset to 5) is available in six positions of the accumulating dial.
3. Dials can be cleared manually when locked.
4. Readily continues a halted division.

5. Positive lock for entire keyboard. (When locked, dividend entry is suppressed.)
6. Has a bell.
7. Automatic transfer from accumulating to counting dial.

Friden disadvantages:

1. Dividend tab key always produces long carriage travel.
2. Divide keys are arranged so that *both* must be depressed for normal (positive) division.
3. Plus and minus bars are located outside the shift keys.
4. The machine is not always ready to use as an adding machine.
5. Unless some dividend tab key is down, dividend will not enter.

Monroe advantages:

1. Lightweight, small, and portable.
2. Smaller models can be built into a full size machine.
3. Smaller models (non-automatic multiplication) have a single key to zero the entire machine.
4. Drive shaft is available for hand cranking or minor unjamming.
5. Factor is entered only once for squaring.
6. Rounding available (one position only).
7. Has back transfer from the accumulating dial to the multiplier unit.
8. A unit counter is available (optional) to tally the number of operations.
9. Single digits of a constant multiplier can be corrected or changed.
10. Zeros in the multiplier are not entered by key depression.
11. Has automatic carriage return to selected position after multiplication.

Monroe disadvantages:

1. Possible runaway on division (lacks divisor lineup feature).
2. Lacks positive keyboard lock.
3. Very easy to depress two keys in one bank at once.

Marchant advantages:

1. Automatic return of carriage after division and instant dividend/divisor entry.
2. Visible keyboard entry dials.
3. Multiplication can be left to right or right to left.
4. Multiplication takes place as the multiplier digits are entered; the product is available just after the last digit of the multiplier is entered.
5. Repeat addition or subtraction takes place if the plus bar is depressed simultaneously with a multiplier key.
6. Can have just one key per bank down at a time.
7. Allows a certain amount of touch control in multiplication with varying concavities of the multiplier keys.

Marchant disadvantages:

1. Lacks auto dividend entry, but see No. 1 advantage. Generally has to be set up for the initial division, but subsequent divisions are programmed faster.

2. Lacks non-entry of the multiplier.
3. Lacks provision for constant multiplier.
4. Lacks dial locks. This makes sums of quotients rather difficult and prevents all operations which depend on split dial locks, such as decimal accumulation.
5. Lacks positive keyboard lock.
6. Algebraic sums of products involves changing controls when the sign changes. (Friden and Monroe have a negative accumulative multiplication key).
7. Lacks automatic dial clearance at the start of a multiplication.

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BIBLIOGRAPHY Z

1048. N. L. FRITZ, "Analog computers for coordinate transformation," *Rev. Sci. Instruments*, v. 23, 1952, p. 667–671.

This is a device for forming a linear combination $a_1b_1 + a_2b_2 + a_3b_3$ by means of an a.c. voltage source and ten turn potentiometers. The answer is obtained by a servo balancer. The effect of load is discussed.

F. J. M.

1049. G. W. SWENSON & T. J. HIGGINS, "Direct current network analyzer for solving wave equation boundary value problems," *Jn. Applied Physics*, v. 23, 1952, p. 126–131.

The two dimensional wave equation $\nabla^2\varphi + (\omega/c)^2\varphi = 0$ is solved for its least characteristic frequency by means of an electrical network. The corresponding difference equations lead to a square lattice network with boundary points at zero potential and with interior points connected to ground through "negative resistors." These "negative resistors" involve an adjustable current source and a galvanometer in a Wheatstone bridge circuit. They are current generators which are adjusted until the output current equals in absolute value the current that would have flowed in a certain resistance connected to ground. However this generated current is opposite in direction to the last mentioned current. In each solution, all these negative resistances are readjusted until all the galvanometers read zero. In order to get a non-zero solution a forcing current is introduced at one point in the network and a solution for the difference equation for the above differential equation obtained. The frequency ω is given an increasing sequence of values starting with zero and the amount of forcing current is noted. As ω approaches the first characteristic frequency, the forcing current approaches zero for a given amplitude for the solution. Convergence of the iteration process occurs past the first characteristic frequency to a point "just below the first antiresonant frequency" and no further. The device will also solve problems in forced vibrations. Various vibration problems are illustrated and the possibility of generalizing the boundary conditions is discussed.

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NOTES

152.—Asymptotic Formulas for the Hermite Polynomials. Asymptotic formulas for the Hermite polynomials $H_n(x)$ are derived by the method of Liouville-Steklov from the integral equation

$$\exp(-\tfrac{1}{2}x^2)\,H_n(x) = \lambda_n \cos(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi)$$
$$+ N^{-\frac{1}{2}} \int_0^x t^2 \exp(-\tfrac{1}{2}t^2)\,H_n(t) \sin(N^{\frac{1}{2}}[x - t])dt$$

where

$$H_n(x) = (-1)^n \exp(x^2) \left(\frac{d}{dx}\right)^n \exp(-x^2), \quad N = 2n + 1$$

and

$$\lambda_n = \begin{cases} |H_n(0)| = n!/(\tfrac{1}{2}n)! & \text{if } n \text{ is even} \\ |N^{-\frac{1}{2}}H_n'(0)| = N^{-\frac{1}{2}}(n+1)!/((n+1)/2)! & \text{if } n \text{ is odd.} \end{cases}$$

The method is an iterative process where one obtains the k-th approximation by placing the $(k-1)$-th approximation under the integral sign to obtain a series in powers of $N^{-\frac{1}{2}}$ which is quite accurate when n is both large and large in comparison with x. Szegö[1] and others do not give the actual coefficients in the asymptotic series, but content themselves with a proof of the existence of the expansions and mention of only the first term or so. Each complete iteration in the process becomes progressively more tedious, and yields only one more term in the expansion. The writer felt that it would be useful to have the explicit expressions for the asymptotic formulas considerably beyond the first term. The expansions which are given below are the result of 9 iterations for both n even and n odd.

These formulas were tested to calculate $H_{19}(x)$ and $H_{20}(x)$ for $x = 1(1)5$ and they permitted a relative error of about 10^{-7} at $x = 1$ which increased steadily with x to about 10^{-2} at $x = 5$.

These formulas can also be used to compute the earlier zeros of $H_n(x)$ by an iterative process.

From the definitions of $A(x)$ and $B(x)$ below, we have

$$0 = A(x) + N^{-\frac{1}{2}} \tan(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi)B(x).$$

We then employ the iteration formula

$$x_{i+1} = N^{-\frac{1}{2}} \{(\tfrac{1}{2}n + m)\pi - \arctan(N^{\frac{1}{2}}A(x_i)/B(x_i))\}$$

where m is a suitably chosen integer (positive, negative, or zero). For the r-th zero of $H_n(x)$ we may use as initial approximation

$$x_0 = \begin{cases} N^{-\frac{1}{2}}(r - \tfrac{1}{2})\pi & n \text{ even} \\ N^{-\frac{1}{2}}(r - 1)\pi & n \text{ odd.} \end{cases}$$

When applied to H_{19} and H_{20} this process gave a relative error of about 10^{-9} for the first two zeros and about 10^{-3} for the last two.

The asymptotic formulas in question may be given as follows.

$$\exp(-\tfrac{1}{2}x^2)\, H_n(x)/\lambda_n = A(x)\cos(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi) + B(x)\, N^{-\frac{1}{2}}\sin(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi)$$

where

$$A(x) = A_0(x) + N^{-1}A_1(x) + \ldots + N^{-4}A_4(x) + O(n^{-5})$$
$$B(x) = B_0(x) + N^{-1}B_1(x) + \ldots + N^{-4}B_4(x) + O(n^{-5}).$$

The polynomials $A_i(x)$, $B_i(x)$ differ slightly in the two cases:

Case I *n even*

$$\begin{aligned}
A_0(x) &= 1\\
A_1(x) &= -x^6/72 + x^2/4\\
A_2(x) &= x^{12}/31104 - 11x^8/1440 + 19x^4/96\\
A_3(x) &= -x^{18}/33592320 + 17x^{14}/622080 - 18889x^{10}/3628800\\
&\quad + 1091x^6/5760 - 19x^2/32\\
A_4(x) &= x^{24}/67722117120 - 23x^{20}/671846400 + 11153x^{16}/522547200\\
&\quad - 177127x^{12}/43545600 + 14601x^8/71680 - 631x^4/384\\
B_0(x) &= x^3/6\\
B_1(x) &= -x^9/1296 + x^5/15 - x/4\\
B_2(x) &= x^{15}/933120 - 7x^{11}/12960 + 901x^7/20160 - 19x^3/48\\
B_3(x) &= -x^{21}/1410877440 + x^{17}/933120 - 2131x^{13}/5443200\\
&\quad + 4421x^9/120960 - 241x^5/384 + 19x/32\\
B_4(x) &= x^{27}/3656994324480 - 13x^{23}/14108774400\\
&\quad + 8503x^{19}/9405849600 - 400187x^{15}/1306368000\\
&\quad + 21500581x^{11}/638668800 - 7159x^7/7680 + 631x^3/192
\end{aligned}$$

Case II *n odd*

$$\begin{aligned}
A_0(x) &= 1\\
A_1(x) &= -x^6/72 + x^2/4\\
A_2(x) &= x^{12}/31104 - 11x^8/1440 + 19x^4/96 - 1/4\\
A_3(x) &= -x^{18}/33592320 + 17x^{14}/622080 - 18889x^{10}/3628800\\
&\quad + 1111x^6/5760 - 21x^2/32\\
A_4(x) &= x^{24}/67722117120 - 23x^{20}/671846400 + 11153x^{16}/522547200\\
&\quad - 59159x^{12}/14515200 + 132641x^8/645120 - 325x^4/192\\
&\quad + 21/32\\
B_0(x) &= x^3/6\\
B_1(x) &= -x^9/1296 + x^5/15 - x/4\\
B_2(x) &= x^{15}/933120 - 7x^{11}/12960 + 901x^7/20160 - 7x^3/16\\
B_3(x) &= -x^{21}/1410877440 + x^{17}/933120 - 2131x^{13}/5443200\\
&\quad + 13333x^9/362880 - 1237x^5/1920 + 21x/32\\
B_4(x) &= x^{27}/3656994324480 - 13x^{23}/14108774400\\
&\quad + 8503x^{19}/9405849600 - 400537x^{15}/1306368000\\
&\quad + 7195607x^{11}/212889600 - 152141x^7/161280 + 671x^3/192.
\end{aligned}$$

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[1] G. SZEGÖ, *Orthogonal Polynomials*. Amer. Math. Soc. *Colloq. Publications*, v. 23, 1939, p. 212–213.

153.—Analytical Approximations, [See also Note 143]. The following twenty-two approximations are all concerned with the so-called offset circle probability function defined by the integral

$$q(R,x) = \int_R^\infty e^{-\frac{1}{2}(\rho^2+x^2)} I_0(\rho x)\rho\, d\rho,$$

in which $I_0(Z)$ is the usual Bessel function. Among other properties, $q(R,x) + q(x,R) = 1 + e^{-\frac{1}{2}(R^2+x^2)} I_0(Rx)$.

The $q(R,x)$ function has been finely tabulated to 6D by the joint effort of NBSINA and RAND.

(13) To better than .0028 over $-\infty \leqslant x \leqslant \infty$,
$$\lim_{R\to 0} \frac{1 - q(R,x)}{1 - q(R,0)} = e^{-\frac{1}{2}x^2} \doteq \frac{1}{(1 + .123x^2 + .010x^4)^4}.$$

(14) To better than .0014 over $-\infty \leqslant x \leqslant \infty$,
$q(1,x) \doteq 1 - .393(1 + .093x^2 + .007x^4)^{-4}$.

(15) To better than .00014 over $-\infty \leqslant x \leqslant \infty$,
$q(1,x) \doteq 1 - .3935(1 + .0968x^2 + .0047x^4 + .00028x^6)^{-4}$.

(16) To better than .0035 over $-\infty \leqslant x \leqslant \infty$,
$q(2,x) \doteq 1 - .865(1+.038x^2 + .004x^4)^{-4}$.

(17) To better than .001 over $-\infty \leqslant x \leqslant \infty$,
$q(2,x) \doteq 1 - .865(1 + .0401x^2 + .00309x^4 + .000075x^6)^{-4}$.

(18) To better than .006 over $0 \leqslant y \leqslant \infty$,
$$\lim_{R\to 0} \frac{1 - q(R,R + y)}{1 - q(R,R)} = e^{-\frac{1}{2}y^2} \doteq \frac{1}{(1 + .015y + .076y^2 + .040y^3)^4}.$$

(19) To better than .00037 over $0 \leqslant y \leqslant \infty$,
$q(.5,.5 + y) \doteq 1 - .1045(1 + .129y + .079y^2 + .056y^3)^{-4}$.

(20) To better than .0007 over $0 \leqslant y \leqslant \infty$,
$q(1,1 + y) \doteq 1 - .267(1 + .203y + .079y^2 + .062y^3)^{-4}$.

(21) To better than .0009 over $0 \leqslant y \leqslant \infty$,
$q(2,2 + y) \doteq 1 - .397(1 + .236y + .066y^2 + .066y^3)^{-4}$.

(22) To better than .0011 over $0 \leqslant y \leqslant \infty$,
$q(4,4 + y) \doteq 1 - .45(1 + .227y + .064y^2 + .065y^3)^{-4}$.

(23) To better than .0013 over $0 \leqslant y \leqslant \infty$,
$$\lim_{R\to\infty} q(R,R+y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \doteq 1 - \frac{.5}{(1+.209y+.061y^2+.062y^3)^4}.$$

(24) To better than .00011 over $0 \leqslant x \leqslant 1$,
$q(1,x) \doteq .6066 + .1500x^2 - .0238x^4$.

(25) To better than .0017 over $0 \leqslant x \leqslant 2$,
$q(2,x) \doteq .135 + .566(x/2)^2 - .096(x/2)^4$.

(26) To better than .0008 over $0 \leqslant x \leqslant 3$,
$q(3,x) \doteq .011 + .231(x/3)^2 + .654(x/3)^4 - .329(x/3)^6$.

(27) To better than .0019 over $0 \leqslant x \leqslant 3$,
$q(3,x) \doteq [.105 + .930(x/3)^2 - .282(x/3)^4]^2$.

(28) To better than .0011 over $0 \leqslant x \leqslant 3$,
$q(3,x) \doteq [.105 + .954(x/3)^2 - .349(x/3)^4 + .043(x/3)^6]^2$.

(29) To better than .002 over $0 \leqslant x \leqslant 4$,
$q(4,x) \doteq [.018 + .581(x/4)^2 + .515(x/4)^4 - .372(x/4)^6]^2$.

(30) To better than .0035 over $0 \leqslant y \leqslant 3$,
$q(3,3 - y) \doteq .568(1 + .157y + .107y^2 + .017y^3)^{-4}$.

(31) To better than .0013 over $0 \leqslant y \leqslant 4$,
$q(4,4-y) \doteq .551(1+.187y+.055y^2+.051y^3)^{-4}$.

(32) To better than .0013 over $0 \leqslant y \leqslant \infty$,

$$\lim_{R\to\infty} q(R,R-y) = \int_{-\infty}^{-y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \doteq \frac{.5}{(1+.209y+.061y^2+.062y^3)^4}.$$

(33) To better than .0004 over $0 \leqslant R \leqslant 1$,
$q(R,R) \doteq 1 - .4921R^2 + .3212R^4 - .0966R^6$.

(34) To better than .0011 over $1 \leqslant R \leqslant \infty$,
$q(R,R) \doteq (R+.183)/(2R-.388)$.

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